Rao-Blackwell

Theorem 1.1 *Let $\mathbf{X} \sim f_{\mathbf{X}}(\mathbf{x}, \theta)$ and T be sufficient for θ, $\mathbf{x} \in \mathfrak{X}$ and $t \in \mathfrak{T}$. Let U be any unbiased estimator for $g(\theta)$. Define $V_t = \mathrm{E}(U|T=t)$. Then V is an unbiased estimator for $g(\theta)$ and $\mathrm{Var}(V) \leq \mathrm{Var}(U)$ with equality iff $V = U$ with probability one.*

Proof 1.1 *Since $U = U(\mathbf{X})$ is an estimator, it is also a statistic. And, since T is sufficient for θ we have*

$$\begin{aligned} V &= \mathrm{E}(U|T=t) && (1)\\ &= \int_{\mathfrak{T}} u(x) f_{X|T}(x|T=t)\ dx && (2) \end{aligned}$$

By Fisher, and noting that $u(x)$ is a function of x and not θ, we see that V is θ-free. Thus, V is a statistic as well.

Further,

$$\begin{aligned} \mathrm{E}(U) &= g(\theta) && (3)\\ &= \int_{\mathfrak{X}} u(x) f_X(x,\theta)\ dx && (4)\\ &= \int_{\mathfrak{T}} \left[\int_{X \ni T=t} u(x) f_{X|T}(x|T=t)\ dx\right] f_T(t,\theta)\ dt && (5)\\ &= \int_{\mathfrak{T}} v(t) f_T(t,\theta)\ dt && (6)\\ &= \mathrm{E}(V) && (7) \end{aligned}$$

So, V is unbiased.

Now,

$$\begin{aligned} \mathrm{Var}(U) &= \mathrm{E}\left(U - \mathrm{E}(U)\right)^2 && (8)\\ &= \mathrm{E}\left(U - \mathrm{E}(V)\right)^2 && (9)\\ &= \mathrm{E}\left((U-V)^2\right) + \mathrm{E}\left((V-\mathrm{E}(V))^2\right) + 2\mathrm{E}\left((U-V)(V-\mathrm{E}(V))\right) && (10) \end{aligned}$$

Since we know that $\mathrm{E}(U) = \mathrm{E}(V)$ by above,

$$\begin{aligned} \mathrm{E}\left((U-V)(V-\mathrm{E}(V))\right) &= \int_{\mathfrak{X}} (V-\mathrm{E}(V))(U-V) f_X(x,\theta)\ dx && (11)\\ &= \int_{\mathfrak{T}} (V-\mathrm{E}(V)) \left[\int_{X \ni T=t} (U-V) f_{X|T}(x|T=t)\ dx\right] f_T(t,\theta)\ dt && (12)\\ &= \int_{\mathfrak{T}} (V-\mathrm{E}(V))[0] f_T(t,\theta)\ dt && (13)\\ &= 0 && (14) \end{aligned}$$

and thus

$$\begin{aligned} \mathrm{Var}(U) &= \mathrm{E}\left((U-V)^2\right) + \mathrm{E}\left((V-\mathrm{E}(V))^2\right) && (15)\\ &\geq \mathrm{E}\left((V-\mathrm{E}(V))^2\right) && (16)\\ &\geq \mathrm{Var}(V) && (17) \end{aligned}$$

with equality iff $\mathrm{E}\left((U-V)^2\right) = 0$ or $V = U$ with probability one.

Example 1.1 *Let* $X_i \overset{iid}{\sim} N\left(\mu, \sigma_0^2\right)$ *so that* $\theta = \mu$. *By exponential family we see that* $T = \sum_{i=1}^n X_i$ *is min suff for* $\theta = \mu$.

Let $U = X_1$ *with* $\mathrm{E}(U) = \mathrm{E}(X_1) = \mu$. *Thus,* U *is an unbiased estimator for* θ *with variance* $\mathrm{Var}(U) = \mathrm{Var}(X_1) = \sigma_0^2$.

Note that $T = \sum_{i=1}^n X_i = U + \sum_{i=2}^n X_i$ *and that*

$$f_{U|T}(u|T=t) = \frac{f_{T|U}(t|U=u)f_U(u)}{f_T(t)}$$

where $U \sim N(\mu, \sigma_0^2)$, $T \sim N(n\mu, n\sigma_0^2)$, *and* $T|U \sim N\left((n-1)\mu + u, (n-1)\sigma_0^2\right)$. *Hence*

$$\begin{aligned}
f_{U|T}(u|T=t) &= \frac{\left(2\pi(n-1)\sigma_0^2\right)^{-1/2} \exp\left(-\frac{(t-(n-1)\mu-u)^2}{2(n-1)\sigma_0^2}\right) \left(2\pi\sigma_0^2\right)^{-1/2} \exp\left(-\frac{(u-\mu)^2}{2\sigma_0^2}\right)}{\left(2\pi n\sigma_0^2\right)^{-1/2} \exp\left(-\frac{(t-n\mu)^2}{2n\sigma_0^2}\right)} && (18)\\
&= \sqrt{\frac{n}{n-1}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{1}{2\sigma_0^2}\left(\frac{(t-(n-1)\mu-u)^2}{n-1} + \frac{(u-\mu)^2}{1} - \frac{(t-n\mu)^2}{n}\right)\right] && (19)\\
&= \sqrt{\frac{n}{n-1}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{1}{2\sigma_0^2}\left(\frac{t^2}{n-1} + (n-1)\mu^2 + \frac{u^2}{n-1} - 2t\mu - \frac{2tu}{n-1} + 2\mu u\right.\right. && (20)\\
&\qquad \left.\left. + u^2 + \mu^2 - 2\mu u - \frac{t^2}{n} - n\mu^2 + 2t\mu\right)\right] && (21)\\
&= \sqrt{\frac{n}{n-1}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{1}{2\sigma_0^2}\left(\frac{u^2}{n-1} + u^2 - \frac{2tu}{n-1} + \frac{t^2}{n-1} - \frac{t^2}{n}\right)\right] && (22)\\
&= \sqrt{\frac{n}{n-1}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{1}{2\sigma_0^2}\left(\frac{n}{n-1}u^2 - \frac{2t}{n-1}u + \frac{t^2}{(n-1)n}\right)\right] && (23)\\
&= \sqrt{\frac{n}{n-1}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{1}{2\sigma_0^2\left(\frac{n-1}{n}\right)}\left(\left(u - \frac{t}{n}\right)^2\right)\right] && (24)
\end{aligned}$$

So, $U|T \sim N\left(\frac{t}{n}, \frac{n-1}{n}\sigma_0^2\right)$ *and thus*

$$\begin{aligned}
V &= \mathrm{E}(U|T=t) && (25)\\
&= \frac{T}{n} && (26)\\
&= \frac{\sum_{i=1}^n X_i}{n} && (27)\\
&= \overline{X} && (28)
\end{aligned}$$

with $\mathrm{E}(V) = \mathrm{E}(\overline{X}) = \mu$ *and* $\mathrm{Var}(V) = \mathrm{Var}(\overline{X}) = \frac{\sigma_0^2}{n}$. *Note that* $\frac{\sigma_0^2}{n} \to 0$ *as* $n \to \infty$ *which is better than* $\mathrm{Var}(U) = \sigma_0^2$ *unless* $n = 1$, *in which case* $V = U$.

Example 1.2 *Let* $X_i \overset{iid}{\sim} U(0, \theta)$. *Lehmann-Scheffe I shows* $T = X_{[n]}$ *is min suff for* θ. *It is also possible to show that* $T = X_{[n]}$ *is complete (enjoy grad school).*

Note that $\mathrm{E}(X_1) = \frac{\theta}{2}$ *so* $\mathrm{E}(2X_1) = \theta$. *Thus,* $U = 2X_1$ *is an unbiased estimator for* θ.

Rao-Blackwell tells us that we need to look at $V = \mathrm{E}(U|T)$. *To do so we need to find the joint density function* $f_{U,T}(u,t)$. *First we note that* $X_1 = X_{[i]}$ *with probability* $\frac{1}{n}$ *for* $i = 1, 2, \ldots,\ n$. *Realizing that we need* $i-1$ *of the* X_i *less than* $X_{[i]}$ *and* $n-i-1$ *of the* X_i *greater than* $X_{[i]}$ *but less than* $X_{[n]}$ *for* $X_i = X_{[i]}$ *we have*

$$f_{X_{[i]},X_{[n]}}(x,y) = \frac{n!}{(i-1)!1!(n-i-1)!1!} \frac{x^{i-1}1(y-x)^{n-i-1}1}{\theta^n} \qquad (29)$$

The univariate distribution of $X_{[n]}$ is found via the CDF and is

$$f_{X_{[n]}}(y) = \frac{n!}{(n-1)!1!}\frac{y^{n-1}}{\theta^n} \tag{30}$$

By conditional probability laws we see that

$$\begin{aligned} f_{X_{[i]}|X_{[n]}}(x|y) &= \frac{(n-1)!}{(i-1)!(n-i-1)!}\frac{x^{i-1}(y-x)^{n-i-1}}{y^{n-1}} && (31)\\ &= \frac{\Gamma(n)}{\Gamma(i)\Gamma(n-i)}\left(\frac{x}{y}\right)^{i-1}\left(\frac{y-x}{y}\right)^{n-i-1}\frac{1}{y} && (32)\\ &= \beta(i,n-i)\left(\frac{x}{y}\right)^{i-1}\left(1-\frac{x}{y}\right)^{n-i-1}\frac{1}{y} && (33)\end{aligned}$$

At this point a little trick helps. Note that $\frac{x}{y} \in (0,1)$. Thus,

$$\int_0^1 \beta(i,n-i)\left(\frac{x}{y}\right)^{i-1}\left(1-\frac{x}{y}\right)^{n-i-1} d\left(\frac{x}{y}\right) = 1 \tag{34}$$

so that

$$\frac{X_{[i]}}{y} \sim Beta(i,n-i) \tag{35}$$

with expectation

$$\begin{aligned} \mathrm{E}\left(\frac{X_{[i]}}{y}\right) &= \frac{i}{i+(n-i)} && (36)\\ &= \frac{i}{n} && (37)\end{aligned}$$

Since y is a constant, this provides $\mathrm{E}\left(X_{[i]}\right) = y\frac{i}{n}$. Substitution allows us to find

$$\begin{aligned} \mathrm{E}\left(2X_{[i]}|X_{[n]} = y\right) &= 2\frac{i}{i+(n-i)}y && (38)\\ &= 2y\frac{i}{n} && (39)\end{aligned}$$

Finally, we can find $\mathrm{E}(U|T=t)$. Since X_1 can be any of the $X_{[i]}$ for $i = 1,2,\ldots,n$ we need to sum over all possibilities. Thus,

$$\begin{aligned} \mathrm{E}\left(2X_1|X_{[n]} = y\right) &= \sum_{i=1}^{n}\frac{1}{n}\left(2y\frac{i}{n}\right) && (40)\\ &= \frac{2y}{n^2}\sum_{i=1}^{n} i && (41)\\ &= \frac{2y}{n^2}\left(\frac{n(n+1)}{2}\right) && (42)\\ &= \frac{n+1}{n}y && (43)\end{aligned}$$

So, $V = \frac{n+1}{n}X_{[n]}$, which is unbiased.