Bootstrap Confidence Intervals

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Bootstrapping.QMD and its output contain information on how to compute/generate bootstrap samples and compute the bootstrap distribution of a statistic. The various approaches that are presented allow the computation of bias and standard error estimates. In what follows, we make use of the **boot** package to fit the bootstrap distributions, and from the distribution a number of different confidence intervals for the parameter of interest.

The boot Package's Bootstrap Distribution

The **boot** package can be obtained from CRAN. Once loaded, it is easy to use the ***boot*** *function* to create bootstrap distributions for a statistic that we have defined. Below we create functions that **boot** can call.;

Estimate $\theta = \mu$ for $N(\mu, \sigma_0^2)$

In Rao-Blackwell and Lehmann-Scheffe we saw that it is possible to update a statistic T to make it unbiased. Consider a n random variables $X_i \overset{iid}{\sim} N(\mu, \sigma_0^2)$ and a statistic $T = \sum_{i=1}^{n} X_i$. Since $E(T) = n\mu \neq \mu$ we wish to find an unbiased estimator.

If we let $U = X_1$ then $E(U) = \mu$ then Rao-Blackwell shows that $V = \overline{X}$ is an unbiased estimator of μ. To confirm this we can look at the bootstrap distributions of T, U, and V. We start by defining a function that computes the statistics within the **boot** function.

```
normal_mystat = function(d, i){
                    n = length(i)     ### Not used and a waste of time

                    T = sum(d[i])     ### Sum_{i=1}^n X_i
                    U = d[i[1]]       ### X_1
                    V = mean(d[i])    ### \overline{X}

                    c(T, U, V)        ### Return the list
                }
```

We generate a number of observations from a normal distribution of our choice.

```
  ### Set the seed to 47 to replicate output.
  set.seed(47)   ### Comment this line out to use the clock.

  n = 100
  mu = 3
  s = 5

  normal_data = rnorm(n, mu, s)

  write.csv(normal_data, "normal_data.csv")
```

We can now use the **normal_mystat** function to get the bootstrap distributions.

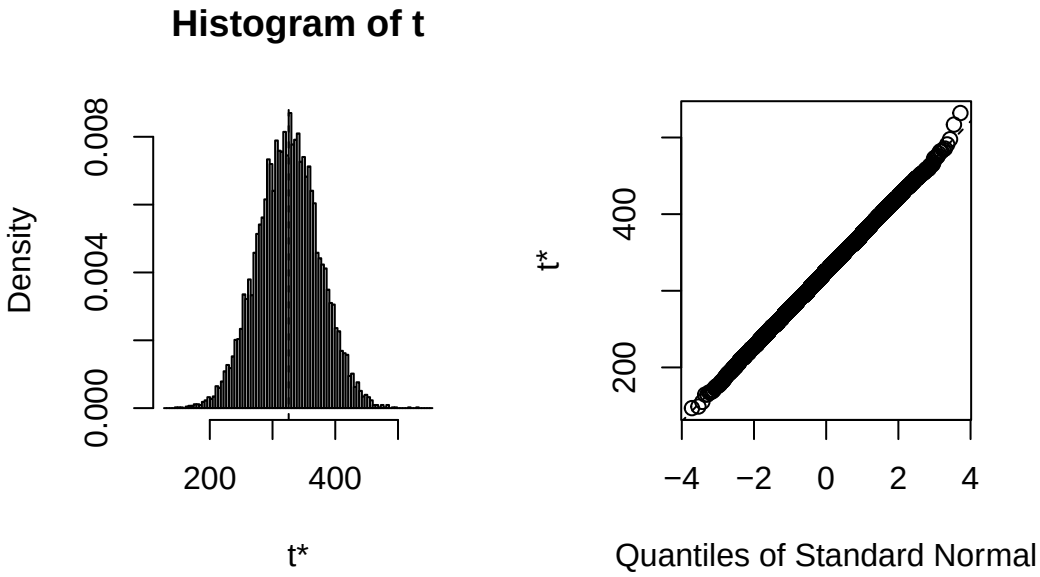
```
  ### Use the boot function to run the bootstrap
  normal_boot = boot(normal_data, normal_mystat, R=9999)

  ### Check the behavior of the statistics
  normal_boot
```

```
ORDINARY NONPARAMETRIC BOOTSTRAP


Call:
boot(data = normal_data, statistic = normal_mystat, R = 9999)


Bootstrap Statistics :
      original       bias    std. error
t1* 325.744525 -0.423393654  48.9468515
t2*  12.973482 -9.701413166   4.8954405
t3*   3.257445 -0.004233937   0.4894685
```

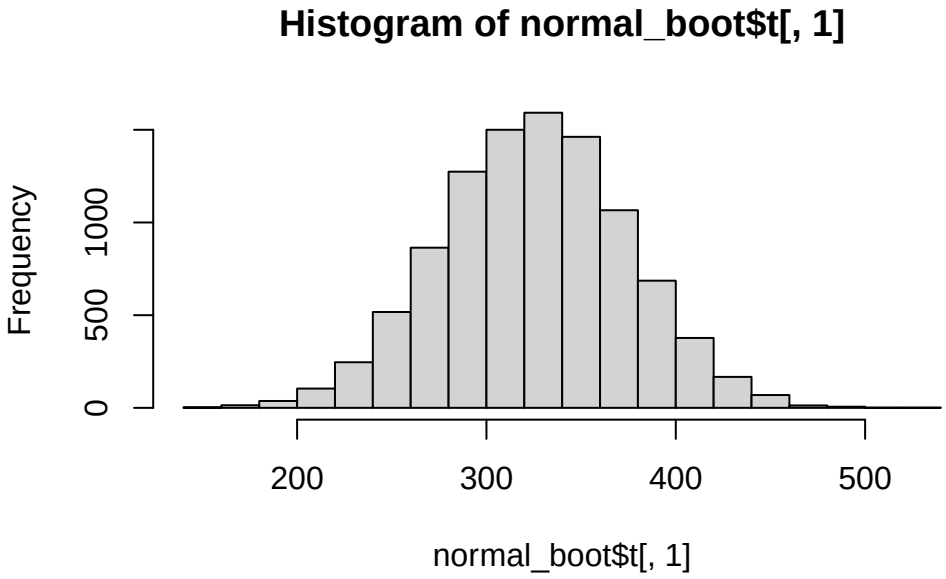
```
  summary(normal_boot)
```

```
Number of bootstrap replications R = 9999
  original   bootBias   bootSE  bootMed
1 325.7445 -0.4233937 48.94685 325.7732
2  12.9735 -9.7014132  4.89544   3.1980
3   3.2574 -0.0042339  0.48947   3.2577
```

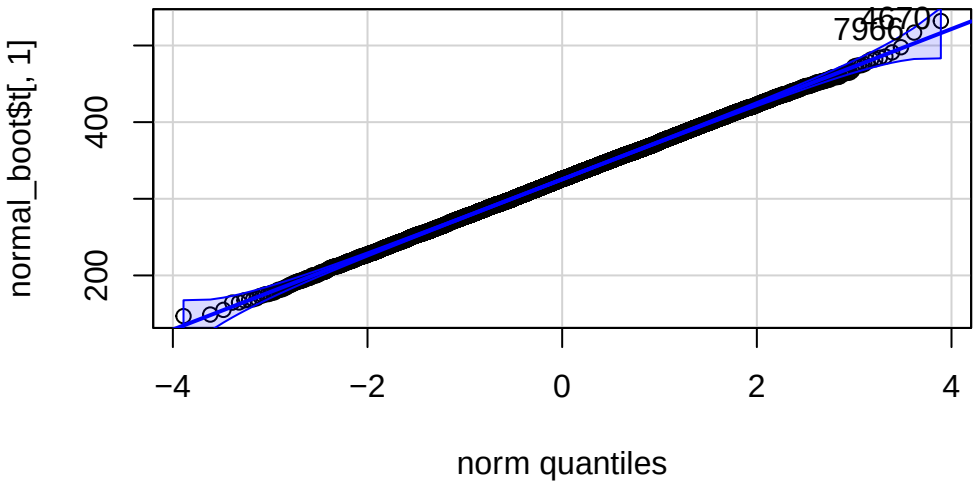
```
  plot(normal_boot)
```




```
hist(normal_boot$t[,1])
```

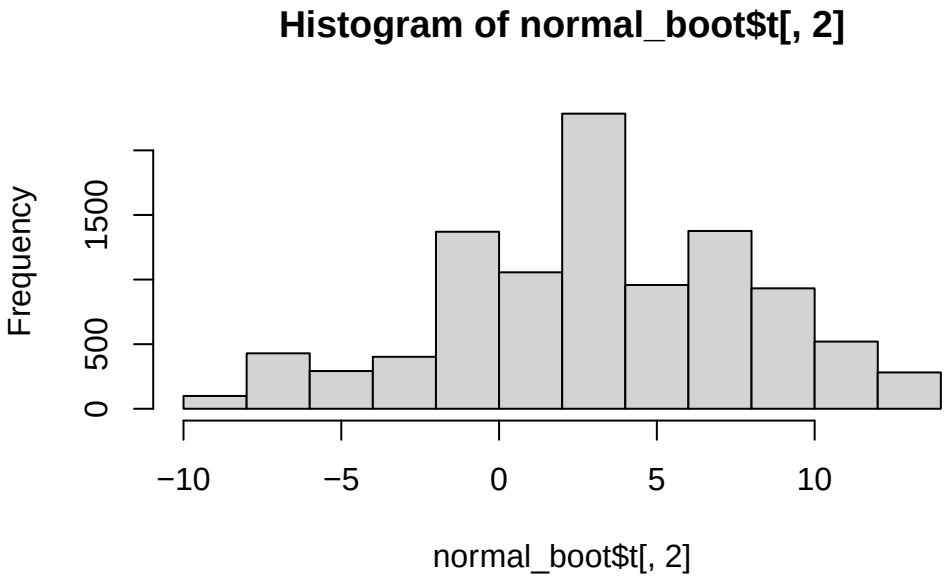



```
qqPlot(normal_boot$t[,1], distribution="norm")
```

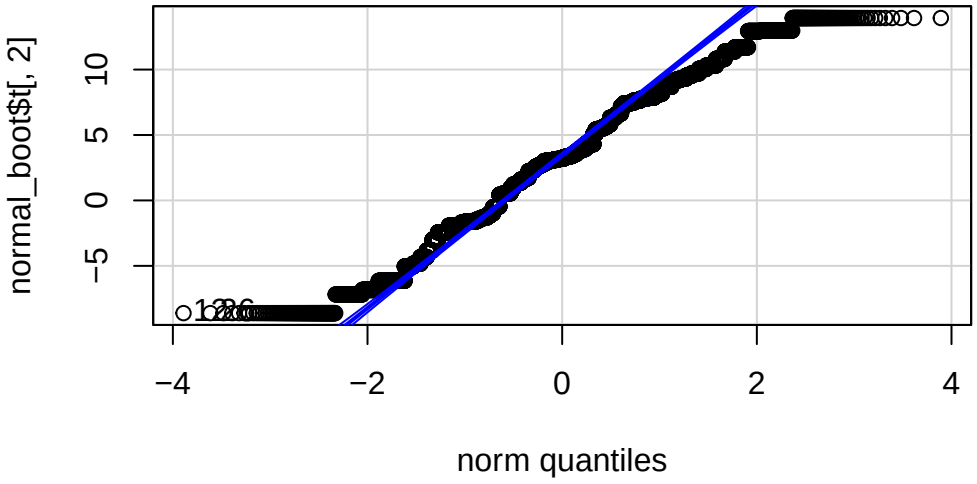



```
[1] 4670 7966
```

```
hist(normal_boot$t[,2])
```

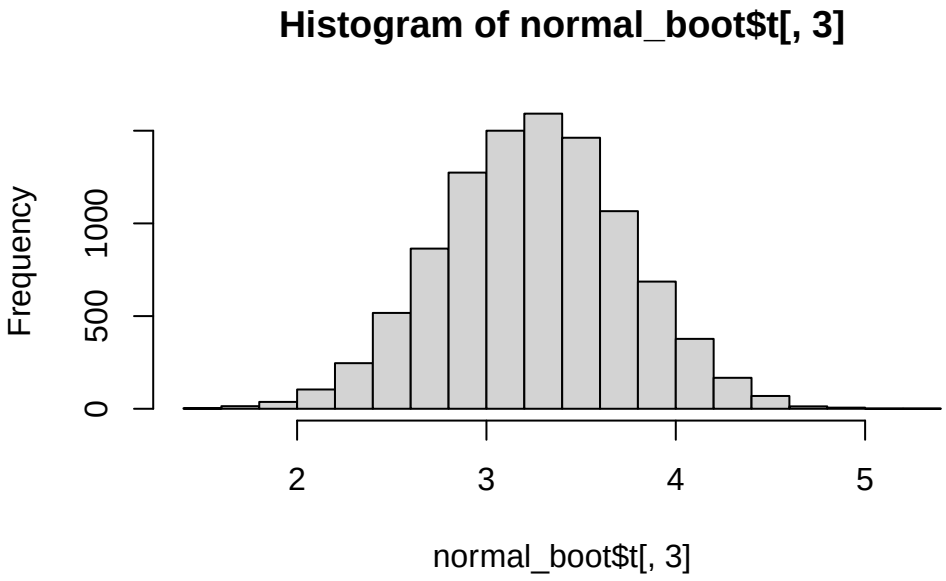



```
qqPlot(normal_boot$t[,2], distribution="norm")
```

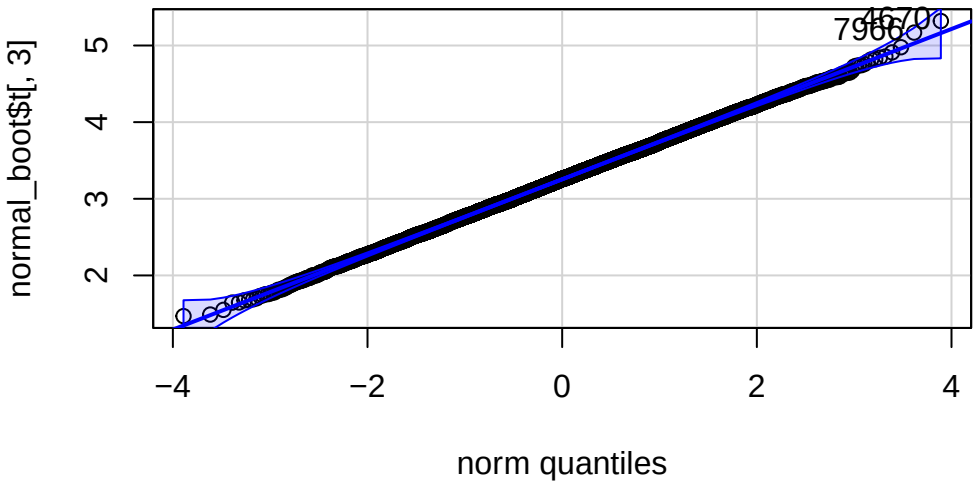



```
[1] 12 26
```

```
hist(normal_boot$t[,3])
```




```
qqPlot(normal_boot$t[,3], distribution="norm")
```




```
[1] 4670 7966
```

Note that the distributions of the sum and mean are both clearly normal. On the other hand, while normal by definition, a single observation appears to be less normal.

```
  quantile(normal_boot$t[,3], c(0.025, 0.975))
```

```
    2.5%    97.5%
2.292393 4.205230
```

```
  args(boot.ci)
```

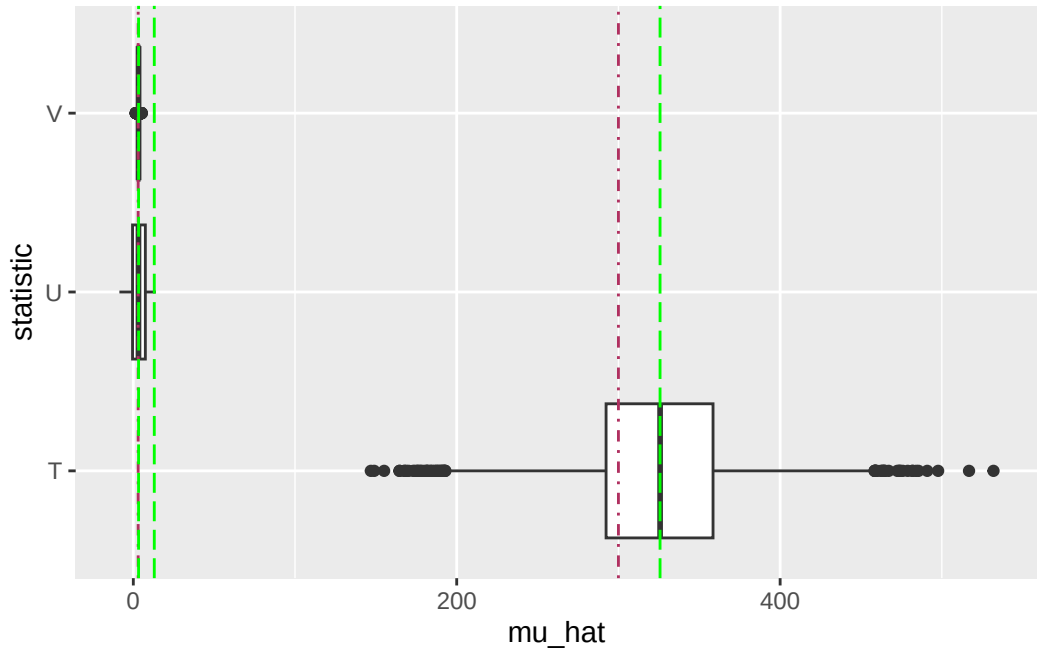
```
function (boot.out, conf = 0.95, type = "all", index = 1L:min(2L,
    length(boot.out$t0)), var.t0 = NULL, var.t = NULL, t0 = NULL,
    t = NULL, L = NULL, h = function(t) t, hdot = function(t) rep(1,
        length(t)), hinv = function(t) t, ...)
NULL
```

```
  boot.ci(normal_boot, type="all", index=3)
```

```
Warning in boot.ci(normal_boot, type = "all", index = 3): bootstrap variances
needed for studentized intervals
```

```
BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 9999 bootstrap replicates

CALL :
boot.ci(boot.out = normal_boot, type = "all", index = 3)

Intervals :
Level      Normal              Basic
95%   ( 2.302,  4.221 )   ( 2.310,  4.223 )

Level     Percentile            BCa
95%   ( 2.292,  4.205 )   ( 2.287,  4.200 )
Calculations and Intervals on Original Scale
```

```r
  x = as.data.frame(normal_boot$t)
  colnames(x) = c("T", "U", "V")
  x |>
      pivot_longer(cols =c("T","U","V"),
                   names_to = "statistic",
                   values_to = "mu_hat",
                   values_drop_na = TRUE) |>
      ggplot(aes(y = statistic, x = mu_hat)) +
        geom_boxplot() +
        geom_vline(xintercept=c(mu, n*mu), color="maroon", lty=4) +
        geom_vline(xintercept=c(normal_boot$t0), color="green", lty=5)
```



We can check that the theoretical and empirical bias and standard errors are similar.

```
  normal_bias = as.data.frame(cbind(summary(normal_boot)$original - c(n*mu, mu, mu),
                                    summary(normal_boot)$bootBias,
                                    c(0,0,0)))
  colnames(normal_bias) = c("Original", "Bootstrap", "Theoretical")
  rownames(normal_bias) = c("Sum","X_1","Xbar")
  normal_bias
```

```
       Original    Bootstrap Theoretical
Sum  25.7445253 -0.423393654           0
X_1   9.9734817 -9.701413166           0
Xbar  0.2574453 -0.004233937           0
```

```
  normal_ses = as.data.frame(cbind(summary(normal_boot)$bootSE, c(sqrt(n)*s, s, s/sqrt(n))))
  colnames(normal_ses) = c("Bootstrap", "Theoretical")
  rownames(normal_ses) = c("Sum","X_1","Xbar")
  normal_ses
```

```
      Bootstrap Theoretical
Sum  48.9468515        50.0
X_1   4.8954405         5.0
Xbar  0.4894685         0.5
```

Note that **boot.ci** uses its own quantile function rather than relying upon the base **quantile** function. Minor interpolation differences between interval estimates are common.

Estimate θ for $U(0, \theta)$

As we saw earlier, Lehmann-Scheffe II can be used to get an estimator for θ when we have n random variables $X_i \overset{iid}{\sim} U(0, \theta)$ and a statistic $T = X_{[n]}$. Since $E(T) = n\theta/(n+1) \neq \theta$ is based upon a minimal sufficient statistic and T can be shown to be complete, we need only to find a constant that makes our new statistic have expectation θ. We note that $V = (n+1)T/n$ has expectation

$$E(V) = E\left(\frac{n+1}{n}T\right) = \frac{n+1}{n}\frac{n}{n+1}\theta = \theta$$

So, V is UMVUE for θ.

We can check the behavior of the unbiased estimators $U = 2\overline{X}$ and $V = (n+1)X_{[n]}/n$ using bootstrapping.

```r
uniform_mystat = function(d, i){
                    n = length(i)               ### Require for V

                    T = max(d[i])               ### X_([n]) is biased
                    U = 2 * mean(d[i])          ### 2 * \overline{X}
                    V = (n+1) * T / n           ### (n+1) X_{[n]} / n

                    c(T, U, V)                  ### Return the list
                }
```

We generate a number of observations from a $U(0, \theta)$ distribution of our choice.

```r
### Set the seed to 47 to replicate output.
set.seed(47)   ### Comment this line out to use the clock.

n = 100
theta = 5

uniform_data = runif(n, 0, theta)

write.csv(uniform_data, "uniform_data.csv")
```

We can now use the **uniform_mystat** function to get the bootstrap distributions.

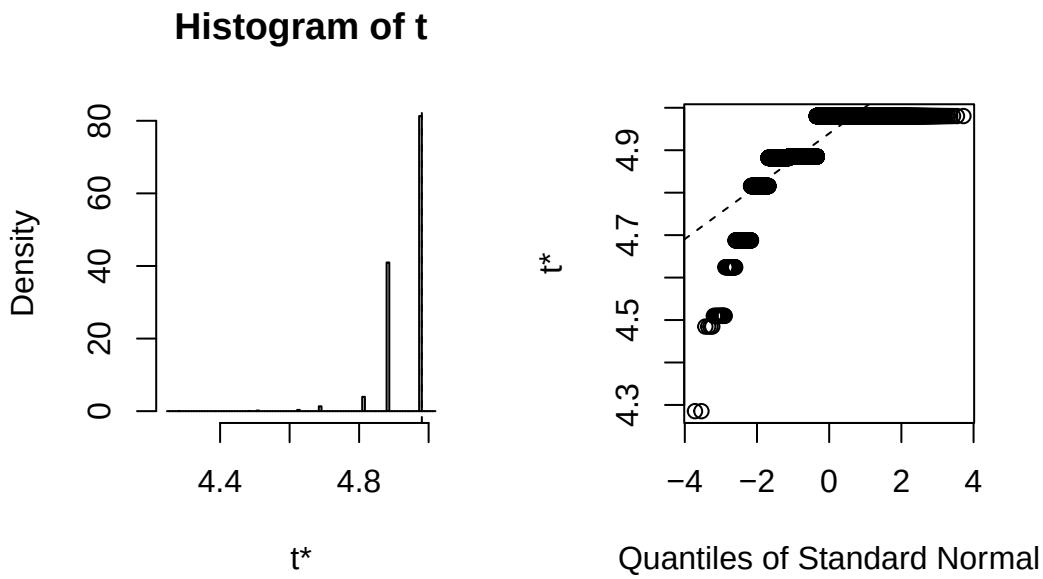
```r
### Use the boot function to run the bootstrap
uniform_boot = boot(uniform_data, uniform_mystat, R=9999)

### Check the behavior of the statistics
uniform_boot
```

```
ORDINARY NONPARAMETRIC BOOTSTRAP


Call:
boot(data = uniform_data, statistic = uniform_mystat, R = 9999)


Bootstrap Statistics :
    original        bias    std. error
t1* 4.980499 -0.0409126659  0.06218675
t2* 4.730612  0.0007536833  0.28421486
t3* 5.030304 -0.0413217926  0.06280862
```

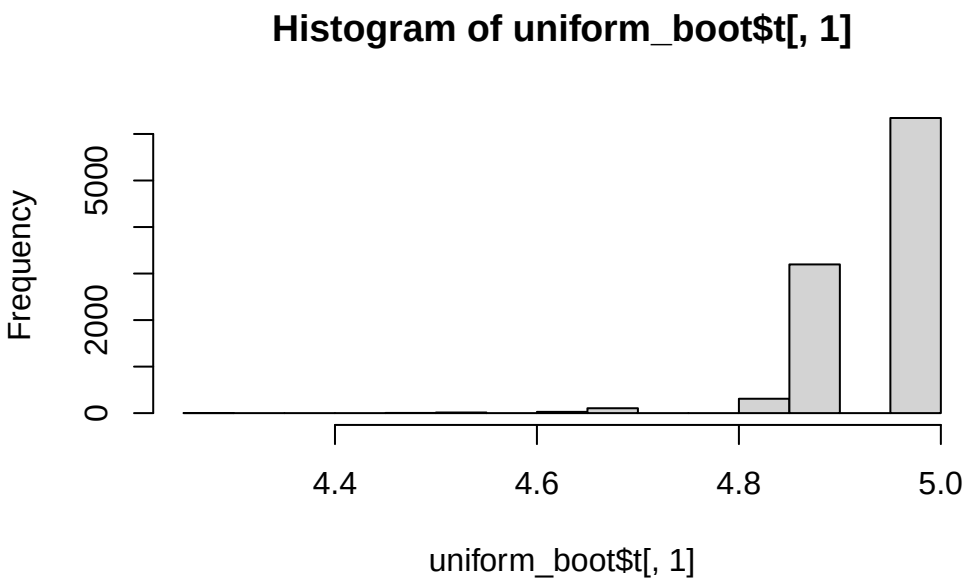
```r
summary(uniform_boot)
```

```
Number of bootstrap replications R = 9999
  original    bootBias   bootSE bootMed
1   4.9805 -0.04091267 0.062187  4.9805
2   4.7306  0.00075368 0.284215  4.7297
3   5.0303 -0.04132179 0.062809  5.0303
```

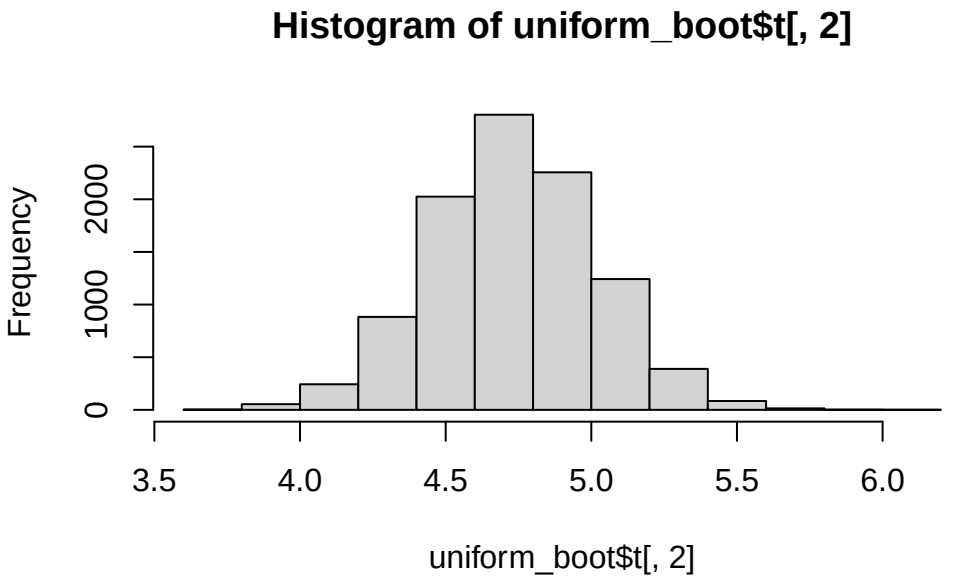
```
plot(uniform_boot)
```



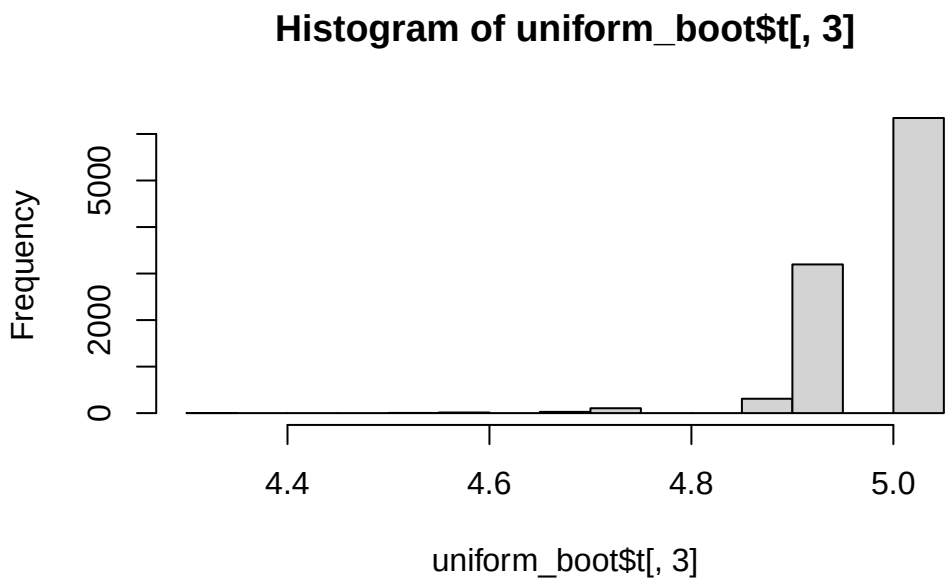

```
hist(uniform_boot$t[,1])
```




```
hist(uniform_boot$t[,2])
```




```
hist(uniform_boot$t[,3])
```



Note that the distributions of the sum and mean are both clearly normal. On the other hand, while normal by definition, a single observation appears to be less normal.

```
  quantile(uniform_boot$t[,3], c(0.025, 0.975))
```

```
    2.5%    97.5%
4.863812 5.030304
```

```
  args(boot.ci)
```

```
function (boot.out, conf = 0.95, type = "all", index = 1L:min(2L,
    length(boot.out$t0)), var.t0 = NULL, var.t = NULL, t0 = NULL,
    t = NULL, L = NULL, h = function(t) t, hdot = function(t) rep(1,
        length(t)), hinv = function(t) t, ...)
NULL
```

```
  uniform_boot.ci = boot.ci(uniform_boot, index=3)
```

```
Warning in boot.ci(uniform_boot, index = 3): bootstrap variances needed for
studentized intervals
```

```
  uniform_boot.ci
```

```
BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 9999 bootstrap replicates
```

```
CALL :
boot.ci(boot.out = uniform_boot, index = 3)

Intervals :
Level      Normal              Basic
95%   ( 4.949,  5.195 )   ( 5.030,  5.197 )

Level     Percentile            BCa
95%   ( 4.864,  5.030 )   ( 4.864,  5.030 )
Calculations and Intervals on Original Scale
```

```r
x = as.data.frame(uniform_boot$t)
colnames(x) = c("T", "U", "V")
x |>
       pivot_longer(cols =c("T","U","V"),
                    names_to = "statistic",
                    values_to = "theta_hat",
                    values_drop_na = TRUE) |>
       ggplot(aes(y = statistic, x = theta_hat)) +
          geom_boxplot() +
          geom_vline(xintercept=theta, color="maroon", lty=2)
```