Fri., 11/22	*(C 14)* **4.2** Field of Polarized Object	
Mon., 11/25 Wed., 11/27 Thurs. 11/28 Fri., 11/29	*(C14)* **4.3** Electric Displacement **Thanksgiving Break** **Thanksgiving** **Thanksgiving Break**	
Mon., 12/2 Wed., 12/4	*(C14)* **4.4.1** Linear Dielectrics (read rest at your discretion) **6.1** Magnetization	HW10

Last Time

Induced Dipoles.

Polarizability *Tensor* $\vec{p} = \vec{\alpha}\vec{E}$

Potential *due to* Dipole $V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2}$ (from Ch. 3)

Field *due to* Dipole $\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$ (from Ch. 3)

Rotating, stretching, and moving Dipoles

$\vec{N} = \vec{p}\times\vec{E}$. $U(\vec{r}) = -\vec{p}\cdot\vec{E}\Big|_{\vec{p}_i}^{\vec{p}_f}$ (for stretching or rotating) $\vec{F} = \left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}$

Summary

Polarization

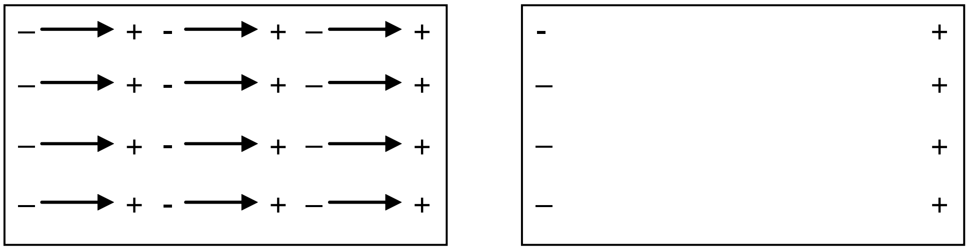
Now we want to move from considering just one dipole to a whole distribution of dipoles. Just as it’s convenient talking about a charge density, and then summing over volume (rather than talking about individual charges and summing over them), it’s convenient to talk about a *dipole density*, a.k.a., the “polarization.”

$\vec{P} = \text{dipole moment per volume}$, $\vec{P} = \frac{d\vec{p}}{d\tau}$ (this is generally, a function of location)

which may be induced by an external electric field or “frozen in.”

Bound Charges (Conceptually)

Surface Charge. Suppose and object has a uniform polarization (density of dipoles). This is equivalent to just having charges on the surfaces (see diagrams below).

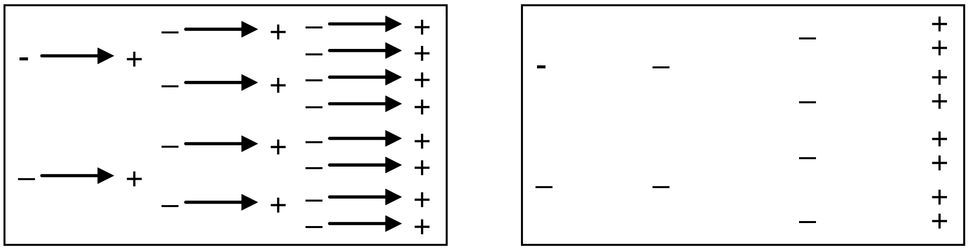


Of course, the scalar potential due to charges distributed over a *surface* is

$$V_{surf}(\vec{r}) = \tfrac{1}{4\pi\varepsilon_o}\oint \frac{\sigma(\vec{r}\,')da'}{\mathscr{r}}$$

Volume Charge

Suppose the polarization changes with position. This is equivalent to *also* having some charges in the volume of the object (see diagrams below).



Of course, the potential at some observation location due to charges distributed through a volume is

$$V_{vol}(\vec{r}) = \tfrac{1}{4\pi\varepsilon_o}\oint \frac{\rho(\vec{r}\,')d\tau'}{\mathscr{r}}$$

The net excess charges due to polarization are called *bound charges,* because they are part of neutral atoms or molecules that are *bound* in place. Any other charge is called *free charge* since they're free to move about.

So, generally, the potential due to a polarized object is (with *b* denoting "bound")

$$V(\vec{r}) = V_{surf}(\vec{r}) + V_{vol}(\vec{r}) = \tfrac{1}{4\pi\varepsilon_o}\oint \frac{\sigma_b(\vec{r}\,')da'}{\mathscr{r}} + \tfrac{1}{4\pi\varepsilon_o}\oint \frac{\rho_b(\vec{r}\,')d\tau'}{\mathscr{r}}$$

Bound Charges (Mathematically)

Now, that makes reasonable *sense*, but it remains to relate these charge densities to the dipole density – Polarization. So, now we'll make that connection.

In chapter 3 we'd derived the dipole term in the multipole expansion to be

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\vec{p}(0)}{r^2}$$

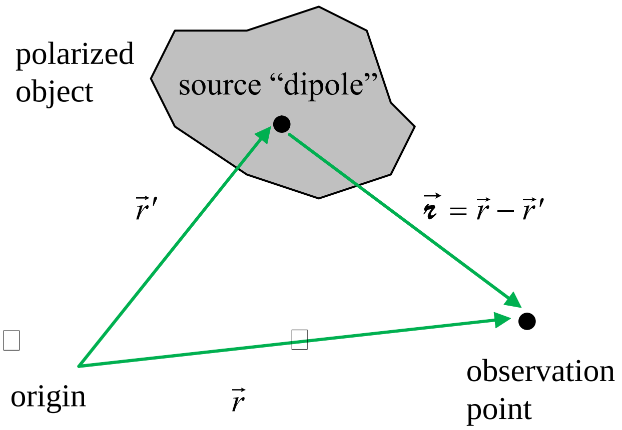
Where I'm explicitly noting that we calculated the dipole relative to the origin. Now, imagine doing that, and then choosing a new coordinate system in which the dipole is centered a location *r'* from the origin. That then gives us

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathscr{r}}\cdot\vec{p}(\vec{r}\,')}{\mathscr{r}^2}$$

This seems like a reasonable way to approximate the field of a molecule or some such other thing that is awfully small and has a dipole moment.

For a chunk of such molecules (or whatever) the electric potential due to the whole collection is

$$V(\vec{r}) = \sum_{dipoles}\frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathscr{r}}\cdot\vec{p}(\vec{r}\,')}{\mathscr{r}^2}$$



Rephrasing this in terms of the dipole density, a.k.a., the polarization, $\vec{P}$, the electric potential is

$$V = \sum_{dipoles}\frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathscr{r}}\cdot\vec{p}(\vec{r}\,')}{\mathscr{r}^2} = \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\hat{\mathscr{r}}\cdot\left(\vec{P}(\vec{r}\,')d\tau'\right)}{\mathscr{r}^2}\ .$$

We suggested that having a *divergence* of dipole moments results in bound charge distributed throughout the volume, so following that hunch, we'd like to rephrase this expression in terms of a divergence; that means squeezing a del in there.

The factor of $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ can be rewritten as $\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)$ (the prime there to point out that we *want* to take the gradient with respect to the source locations, not the observation location.)

Note: We've previously seen and used that

$$\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2},$$

so it's probably not too hard to imagine that

$$\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

since

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'.$$

which gives

$$V = \frac{1}{4\pi\varepsilon_0} \int\limits_{volume} \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) \cdot \vec{P}\, d\tau'.$$

Use a product rule to rewrite the integrand:

$$\vec{\nabla}' \cdot \left(\frac{1}{\mathscr{r}}\vec{P}\right) = \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) \cdot \vec{P} + \frac{1}{\mathscr{r}}\left(\vec{\nabla}' \cdot \vec{P}\right),$$

$$\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) \cdot \vec{P} = \vec{\nabla}' \cdot \left(\frac{1}{\mathscr{r}}\vec{P}\right) - \frac{1}{\mathscr{r}}\left(\vec{\nabla}' \cdot \vec{P}\right).$$

The electric potential can be written as the sum of two terms:

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int\limits_{volume} \vec{\nabla}' \cdot \left(\frac{1}{\mathscr{r}}\vec{P}\right) d\tau' - \int\limits_{volume} \frac{1}{\mathscr{r}}\left(\vec{\nabla}' \cdot \vec{P}\right) d\tau'\right].$$

Apply the divergence theorem to the first integral to get

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int\limits_{surface} \frac{1}{\mathscr{r}}\vec{P} \cdot d\vec{a}\,' - \int\limits_{volume} \frac{1}{\mathscr{r}}\left(\vec{\nabla}' \cdot \vec{P}\right) d\tau'\right],$$

This has the form we're looking for. If we identify

$$\sigma_b = \vec{P} \cdot \hat{n} \quad \text{and} \quad \rho_b = -\vec{\nabla} \cdot \vec{P} \qquad \text{(4.11 \& 4.12)}$$

and $\hat{n}$ is a unit vector normal to the surface (pointing outward).

Then we have

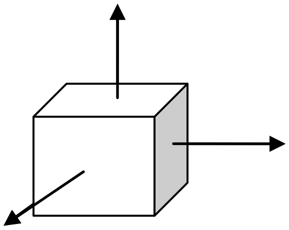
$$V = \frac{1}{4\pi\varepsilon_0}\left[\int\limits_{surface} \frac{\sigma_b da'}{\mathscr{r}} + \int\limits_{volume} \frac{\rho_\text{b}\, d\tau'}{\mathscr{r}}\right], \qquad (4.13)$$

Which is what we said made conceptual sense.

If you want to find the electric field from the polarization $\vec{P}$, there are two options:

1. Use $\vec{P}$ to find V using Equations 4.11-13, then use $\vec{E} = -\vec{\nabla}V$.

2. Find the bound charges σ_b and ρ_b from $\vec{P}$, then use Gauss's law or Coulomb's law to find $\vec{E}$.



Examples/Exercises:

Problem 4.31

A dielectric *cube* of side *a*, centered at the origin, carries a "frozen-in" polarization $\vec{P} = k\vec{r}$, where *k* is a constant. Find all of the bound charges and check that they add up to zero.

The polarization can be written as $\vec{P} = k\vec{r} = k\left(x\,\hat{x} + y\,\hat{y} + z\,\hat{z}\right)$ The volume bound charge is

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) = -\left[\frac{\partial(kx)}{\partial x} + \frac{\partial(ky)}{\partial y} + \frac{\partial(kx)}{\partial z}\right] = -3k,$$

which is uniform throughout the volume. On the top surface, $\hat{n} = \hat{z}$ and $z = a/2$, so the bound surface charge is

$$\sigma_b = \vec{P}\cdot\hat{n} = k\left[x\,\hat{x} + y\,\hat{y} + (a/2)\,\hat{z}\right]\cdot\hat{z} = ka/2,$$

which is uniform over the surface. By symmetry, it is the same on all six faces. The total bound charge is

$$\begin{aligned} Q_b &= \rho_b Vol + \sigma_b A \\ Q_b &= (-3k)a^3 + 6\left[(ka/2)a^2\right] = 0 \end{aligned}.$$

Problem 4.14

When you polarize a neutral dielectric, charges move a bit, but the *total* remains zero. This fact should be reflected in the bound charges σ_b and ρ_b. Prove that the total bound charge vanishes.

The total bound charge is found by integrating σ_b over the surface and ρ_b over the volume:

$$Q_b = \oint_{surface} \sigma_b da + \int_{volume} \rho_b d\tau.$$

Use the definitions of the bound charges to write this in terms of the polarization as

$$Q_b = \oint_{surface} \left(\vec{P}\cdot\hat{n}\right)da + \int_{volume} \left(-\vec{\nabla}\cdot\vec{P}\right)d\tau = \oint_{surface} \vec{P}\cdot d\vec{a} - \int_{volume} \left(\vec{\nabla}\cdot\vec{P}\right)d\tau,$$

because $d\vec{a} = da\,\hat{n}$. The second integral can be rewritten using the divergence theorem:

$$Q_b = \oint_{surface} \vec{P}\cdot d\vec{a} - \oint_{surface} \vec{P}\cdot d\vec{a} = 0.$$

This is, in fact, central to Griffith's derivation of the bound charge volume-density in section 4.2.2

Exercise – have the students try this

A dielectric *cylinder* of radius *R* and length *L* is centered on the *z* axis. One end of the cylinder is at *z* = 0. It carries a "frozen-in" polarization $\vec{P} = k(1 + z/L)\hat{z}$, where *k* is a constant. Find all of the bound charges and check that they add up to zero.

The volume bound charge is

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \right) = -k \frac{\partial}{\partial z}(1 + z/L) = -k/L,$$

which is uniform throughout the volume. On the curved surface, $\hat{n} = \hat{s}$, so the bound surface charge there is

$$\sigma_b = \vec{P} \cdot \hat{n} = \left[k(1 + z/L)\hat{z} \right] \cdot \hat{s} = 0.$$

On the end cap at $z = 0$, $\hat{n} = -\hat{z}$ so the bound surface charge there is

$$\sigma_b = \vec{P} \cdot \hat{n} = (k\,\hat{z}) \cdot (-\hat{z}) = -k.$$

On the end cap at $z = L$, $\hat{n} = \hat{z}$ so the bound surface charge there is

$$\sigma_b = \vec{P} \cdot \hat{n} = (2k\,\hat{z}) \cdot (\hat{z}) = 2k.$$

The total charge is

$$Q_b = (-k/L)(\pi R^2 L) + (-k)(\pi R^2) + (2k)(\pi R^2) = 0.$$

4.2.2 Physical Interpretation of Bound Charges

Through section 4.2.2, Griffith's goes to some length to convince us that, these identifications, $\sigma_b = \vec{P} \cdot \hat{n}$ and $\rho_b = -\vec{\nabla} \cdot \vec{P}$ aren't just conceptual tools for defining *equivalent* charge densities, but they are the *actual* charge densities in a polarized medium.

For the surface charge density, he asks us to imagine a cylinder of aligned dipoles, then the whole cylinder's dipole moment would be (where L is the length of the cylinder and A is the cross-sectional area)

$$\begin{aligned} (P \cdot Vol) &= qL \\ PA_{cross}L &= qL \\ PA_{cross} &= q \end{aligned}$$

So the charge on the face of the cylinder is equal to PA.

Now, if the object is cut obliquely, rather than perpendicularly, what we want is still the *cross-sectional* area, $A_{cross} = A_{end} \cos\theta$

$$PA_{cross} = q$$
$$PA_{end}\cos\theta = q$$

So,
$$P\cos\theta = \frac{q}{A_{end}}$$
$$\vec{P}\cdot\hat{n} = \frac{q}{A_{end}} = \sigma_b$$

As for the volume charge density. Since we're imagining a net neutral object, if we wrap it in a closed Gaussian surface, then whatever charge we catch *inside* the surface must be equal and opposite to the charge *on* the surface. That is,

$$\oint \sigma_b da = -\int \rho_b d\tau$$
$$\oint \vec{P}\cdot d\vec{a} = -\int \rho_b d\tau$$
$$\int \vec{\nabla}\cdot\vec{P} d\tau = -\int \rho_b d\tau$$

(last step uses the Divergence Theorem / Gauss's Law)

This is clearly true over the full volume of the object, but, since dipoles are neutral charge pairs, if you move your Gaussian surface around, it must still be true – what extra charge there is inside must be countered by extra charge on the surface. So this is true for all volumes, i.e. the integrands must be equal.

$$\vec{\nabla}\cdot\vec{P} = -\rho_b$$

Preview

For Monday, you'll read about the electric displacement and Gauss's law in materials.

"why is the field inside a surface uniform?"
Davies

"Can we do a problem where we find the electric filed of a non uniformly polarized volume?"
Jessica

"Why is (- del dot P) a volume charge potential? (pg. 174)"
Casey P,

"I had a little trouble understanding the difference between microscopic and macroscopic electric field."
Connor W,

"Can we talk conceptually about how the bound charges concept and how the surface/volume charge distributions are related to the dipole moments?"
Sam

"I had trouble following Example 4.2 and the different arguments he was making. Could we do this example (or one similar to it)?"
Casey McGrath

"Why is there a volume term and a surface term in equation 4.13 for the potential? Isn't the "surface charge" included in the volume integral?"
Spencer

"What does he mean by bound charges? On page 174, I don't really see how its so enlightening to say that the potential of charges in an area has to do with the surface and the volume inside, I would have thought that was a given."
Freeman,