Numerical Analysis

FINAL EXAM - TAKE HOME - 300 points

Please be very clear in your write up. Explain what you are solving, what methods you are using, why your solutions make sense. Please do your own work, NO TEAM WORK! This should reflect your own individual thinking and process and any suspect work is grounds for failing the class. Each problem is worth 100 points

Problem 1. The differential equation

\[ \frac{dh}{dt} = -0.6\pi r^2 \sqrt{-2g \frac{\sqrt{h}}{A(h)}} \]

where \( A(h) = \pi h^2 \), models liquid flowing from an inverted conical tank with a circular hole in the bottom. In this equation \( r \) is the radius of the hole, \( h \) is the height of the liquid in the tank, measured from the bottom (tip of the cone), and \( A(h) \) is the area of the cross section of the tank \( h \)-units above the hole. Suppose \( r = 0.2 \text{ ft} \), \( g = -32.17 \text{ ft/s}^2 \) and that initially the water level \( h = h_0 \).

1. Draw a sketch of the problem.
2. Are there any values of \( h_0 \) for which the differential equation might have existence or uniqueness problems? Explain why this might make physical sense.
3. Choose a numerical method to solve this problem. State what method your are using and explain how it was derived and the error associated with the method.
4. Do you expect any issues with the convergence of your method?
5. Use your numerical method to compute the water level after 10 minutes if \( h_0 = 4 \).
6. Determine approximately, when the tank will be empty.

In your write-up please include the extremely well commented code that you wrote using the verbatim command. You should make sure to clearly answer the questions above.

Problem 2. Consider the following interpolation problem: Find values for \( a, b, c, \) and \( d \), for \( y = ax^3 + bx^2 + cx + d \) to interpolate a cubic polynomial assuming you are given four points \((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\). We are going to use matrix inverse methods to solve for these points rather than our polynomial interpolation formulas.

1. Write down a system of four equations (one for each of the points \((x_i, y_i)\)) for four unknowns \((a, b, c, d)\).
2. State this problem as a matrix problem of the form \( Ax = b \).
3. Discuss the plusses and minuses of using a direct or iterative method on this system of equations, this should include a discussion of convergence for the iterative methods. Which method should you choose?
4. Write code that uses the chosen numerical method (not matrix inverse) to solve the system of equations for \( a, b, c, \) and \( d \). State which method you are using and why.
5. Finally, using the numbers generated from your numerical solution, plot the cubic along with the points to demonstrate the interpolation is working.
6. Test your code using simple values. For example points that you know will give a straight line. Then use the points \((1, 2.7), (1.5, 3), (2, 3.5), (2.5, 4)\).

In your write-up please include the extremely well commented code that you wrote using the verbatim command. You should make sure to clearly answer the questions above. Also write down the interpolating polynomial and show a graph of the function for the points given. How would you generalize this code to interpolate \( n \)-points?
Problem 3.

A planet's movement around the sun can be expressed parametrically, at time $t$, by the equations:

$$x(t) = a (\cos(E(t)) - e)$$
$$y(t) = a \sqrt{1 - e^2 \sin(E(t))}$$

according to Kepler's Laws of planetary motion. Here $a$ is the semi-major axis (for Earth $a = 152,098,232$), $e$ is the eccentricity of the planet's orbit (for Earth $e = 0.0167$), and $E$ is the eccentric anomaly given by

$$E(t) = \omega t + e \sin(E(t))$$

where $\omega$ is the frequency of orbit, in radians, (for Earth this is $\omega = \frac{2\pi}{360.25635}$). We have assumed units for time in days and length in kilometers. We assume that when $t = 0$ the planet is located at the perifocus, where the planet is closest to the sun. Notice that we cannot solve directly for $E$ with an exact formula. This is one of the most famous examples of a nonlinear root finding equation in science. We can however use a root finding method to solve for $E$ at a specific time, $t$, and then plug values into $(x(t), y(t))$ to find the planet's location and plot the orbit around the sun. We are going to write code that solves for and plots the planet's location.

1. First write the equation for $E$ as a root finding problem, $f(E) = 0$.
2. Now choose a root finding method to solve this problem. Discuss why you chose that method. What possible issues might you have?
3. Write code that will solve for the earth's location at any time $t$, using your method to find the root $f(E) = 0$ and then plugging in to $x(t)$ and $y(t)$. Here you will use a for loop though time to calculate the position of the earth for time step. You can use an error tolerance of $10^{-3}$ for this problem.
4. Then plot $x(t)$ vs. $y(t)$ to see the parametric curve of the earth's orbit.

In your write-up please include the extremely well commented code that you wrote using the verbatim command. You should make sure to clearly answer the questions above. You should also include a picture of the planetary orbit and a table of values that gives $t$, $x(t)$ and $y(t)$ at the following times: $t = 0, 100, 200$. 