Imperfect Bifurcation and Catastrophe.

Ex: \( x = h + rx - x^3 \)

\[ \text{this looks like our symmetric pitchfork form.} \]

\[ \text{we add an extra parameter} \]

\[ \text{This breaks the symmetry.} \]

\( h \neq 0 \) then \( h \) is an imperfect parameter.

We will analyze this system for both parameters.

LOTS OF CASES!!

1. Solve for Fixed Points!

\( h + rx - x^3 = 0 \) separate:

\[ h = x^3 - rx \]

**Cases for \( r \):** \( r < 0 \) \( r = 0 \) \( r > 0 \)

- \( r < 0 \)

  \[ \text{let } r = -1 \]

  \( x^3 + x \)

- \( r > 0 \)

  \[ \text{let } r = 1 \]

  \[ \text{More complicated...} \]

For some values of \( h \)

One FP... but some values of \( h \) have

Three FP.
Summarize FP.

**EASY** — when \( r \leq 0 \) One FP. For all values of \( h \).

**HARD** — when \( r > 0 \) more complicated...

Bifurcations happen when the number of FP changes.

How can we solve for the "critical" \( h \) values when \( r > 0 \)?

These happen when \( x^3 - rx \) reaches a local max or min.

From Calculus!

Find max and min of \( x^3 - rx \)

Take derivative — set to zero

\[
\frac{d}{dx}[x^3 - rx] = 3x^2 - r = 0
\]

\( r = 3x^2 \) or when \( x = \pm \sqrt[3]{\frac{r}{3}} \)

Plug into \( h = -rx + x^3 \) to find critical \( h \) values.

\[
h = r\left[\pm \sqrt[3]{\frac{r}{3}}\right] - \left[\pm \sqrt[3]{\frac{r}{3}}\right]^3
\]

\[
= \pm r \sqrt[3]{\frac{r}{3}} \pm \frac{r \sqrt[3]{r}}{3^{\frac{1}{3}}} = \pm \frac{r \sqrt[3]{r} + r \sqrt[3]{r}}{3^{\frac{1}{3}}}
\]

\[
= \pm \frac{r \sqrt[3]{r} (\sqrt[3]{3} - 1)}{3^{\frac{1}{3}}} = \pm \frac{2r \sqrt[3]{r}}{3^{\frac{1}{3}}}
\]

So bifurcation points happen at \( h = \pm \frac{2r \sqrt[3]{r}}{3^{\frac{1}{3}}} \) really this is a bifurcation curve.

\* INTRODUCE STABILITY DIAGRAM! analyze two parameters at one time.
Plot \( h \) vs \( r \) and then write in stability information!

1. Draw \( h = \pm \frac{2r\sqrt{r}}{3} \)

   \[
   \begin{array}{c}
   \text{ONE FIXED POINT} \\
   \text{Below critical } h \\
   \text{THREE FIXED POINTS.}
   \end{array}
   \]

   \[
   \begin{array}{c}
   \text{Above critical } h \\
   \text{ONE FIXED POINT}
   \end{array}
   \]

   Shows the \# of fixed points as we move around parameter space!

So we know the \# of Fixed Points ... what's next?


   If \( h = 0 \) Easy \( x = rx - x^3 \)

   Typical Pitchfork

   \[
   \begin{array}{c}
   \text{Typical Pitchfork}
   \end{array}
   \]

   When \( h \neq 0 \) we have some cases:

   Fixed \( h \): Plot \( x^* \) vs \( r \)

   \( h + rx - x^3 = 0 \) solve for \( r \)

   \[
   r = \frac{x^3 - h}{x}
   \]

   \( \text{asymptote at } x = 0 \)

   \( \text{use the flip trick.} \)
When $h \neq 0$ we have some cases!

**CASE** $r < 0$ let $r = -1$

- Expect one FP for all $h$.

*Flow on line*

- $h < 0$
  - Stable

- $h = 0$
  - Stable

- $h > 0$
  - Stable

So when $r < 0$ we have one Fixed Point and it is stable.

**CASE** $r = 0$

- Still expect one FP for all $h$.

*Flow on line*

- Again we would find always stable.

So when $r = 0$ we have one stable Fixed Point.

**CASE** $r > 0$

- Expect one or three FP. let $r = 1$

Then $h = \pm \frac{2}{3\sqrt{3}} \approx \pm 0.5849$.

- $h > \frac{2}{3\sqrt{3}}$ let $h = 0.5$
  - One stable FP

- $\frac{2}{3\sqrt{3}} > h > -\frac{2}{3\sqrt{3}}$ is the critical value.
  - $h < -\frac{2}{3\sqrt{3}}$ let $h = -0.5$
  - Three FP

- $h > 0$
  - One stable.

So outside of critical range one stable FP

Inside three FP the middle one is unstable.
Put this all together in Bifurcation Diagrams for Positive vs. Negative $h$.

$h > 0$

\[
\text{let } h = 1
\]

\[
\text{plot } r = \frac{x^3 - h}{x}
\]

then FLIP.

$h < 0$

\[
\text{let } h = -1
\]

FLIP

ADD STABILITY

In real physical systems you can usually say $h > 0$ or $h < 0$ only—because one does not make physical sense.

CATASTROPHE.

- Instead of a smooth transition imagine you start at (A) then slowly turn down your parameter $r$.

- At some point you drop off the edge to the lower solution curve and even increasing $r$ at that point will not get you back to the top solution curve.
\[ N = RN(1 - \frac{N}{k}) - \frac{BN^2}{A^2 + N^2} \]

too many parameters to deal with individually!

**NON-DIMENSIONALIZATION** - a trick of applied math!

Goal - define new variables that simplify the equation.

Non-unique - the is more than one correct choice for the variables... in real life we try lots of ways and choose the best one.

1. Define the new variables...

   let \[ x = \frac{N}{A} \] be our new dependent variable.

   and \[ t = \frac{B}{A} \] be our new independent variable.

2. Substitute new variables in one at a time.

   If

   \[ x = \frac{N}{A} \]

   then \[ N = Ax \text{ and } \dot{N} = A \dot{x} \]

   our equation becomes

   \[ Ax = RAX\left(1 - \frac{Ax}{k}\right) - \frac{BA^2x^2}{A^2 + A^2x^2} \]

   \[ Ax = RAX\left(1 - \frac{Ax}{k}\right) - \frac{BX^2}{1 + X^2} \]

   then \[ t = \frac{B}{A} t \] and \[ t = \frac{A}{B} \]

   need to substitute for \[ \frac{d}{dt} = \frac{dx}{dt} \frac{d}{dx} \]

   so \[ \frac{d}{dt} = \frac{B}{A} \frac{d}{dt} \]

   I can sub for \[ x = \frac{dx}{dt} \]

   \[ A \frac{B}{A} \frac{dx}{dt} = RAX\left(1 - \frac{Ax}{k}\right) - \frac{BX^2}{1 + X^2} \]

simplify this...
\[ \frac{dx}{dt} = \frac{RA}{B} x \left( 1 - \frac{Ax}{k} \right) - \frac{x^2}{1+x^2} \]

3. Define new "simpler" parameters:

\[ r = \frac{RA}{B} \quad \text{and} \quad k = \frac{k}{A} \]

\[ \dot{x} = r x \left( 1 - \frac{x}{k} \right) - \frac{x^2}{1+x^2} \]

analyze this equation in terms of \( r \) and \( k \).