Today we want to analyze **Nonlinear Systems**

\[
\begin{align*}
\dot{x} &= f_1(x, y) \\
\dot{y} &= f_2(x, y)
\end{align*}
\]

**Phase Plane** is helpful... but limited.

**Important Ideas:**
- **Fixed Points** - we always solve for these.
  
  \[
  f_1(x, y) = 0 \\
  f_2(x, y) = 0
  \]

  **Stable, Unstable, Center,**...

  **Sink, Source**

- **Closed Orbits** - orbits that correspond to oscillatory or repeating solutions.

- **The Behavior of FP**... type of fixed point and confirm stability.

**Numerical solutions are possible** - USE RUNGE KUTTA

but only if a **Number** is needed.

**Helpful Idea:** **Nullclines** - lines in the phase plane along which one of \( \dot{x} \) or \( \dot{y} \) is zero — flow in only one direction.

**Example:**

\[
\begin{align*}
\dot{x} &= x + e^{-y} \\
\dot{y} &= -y
\end{align*}
\]

**Fixed Points**

\[
\begin{align*}
x + e^{-y} &= 0 \\
-x &= x + 1 = 0 \\
x &= -1
\end{align*}
\]

\[
\begin{align*}
-x &= 0 \\
y &= 0
\end{align*}
\]

One **FP** at \((-1, 0)\)

**Phase Plane** — This appears to be a saddle point.

**Nullclines:**

\[
\begin{align*}
x + e^{-y} &= 0 \quad \text{or} \quad e^{-y} &= -x \\
-y &= 0 \quad \text{or} \quad y &= 0
\end{align*}
\]

along \( y = 0 \)

along \( x = x + 1 \)

along \( y = \ln(\frac{1}{x}) \)

nullclines intersect @ Fixed Points

**Confirms Saddle...**

**But... I would like more proof!**
Goal: Find Fixed Points — zoom in on them to see what kind of FP's they are.

Consider the system
\[ \dot{x} = f(x, y) \]
\[ \dot{y} = g(x, y) \]

Suppose that \((x^*, y^*)\) is a fixed point.

We want to see what happens right around this point, "flick it and see what it does."

Let \(u = x - x^*\) and \(v = y - y^*\).
\(u\) — flick in \(x\)-direction \(v\) — flick in \(y\)-direction

Now we want to know do these disturbances grow or decay.

grow \(\Rightarrow\) unstable
decay \(\Rightarrow\) stable.

\[ \dot{u} = \frac{d}{dt}(x - x^*) = \dot{x} = f(x^* + u, y^* + v) = f(x^*, y^*) + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + \ldots \]

Two-dimensional Taylor series

\[ \dot{v} = \frac{d}{dt}(y - y^*) = \dot{y} = g(x^* + u, y^* + v) = g(x^*, y^*) + u \frac{\partial g}{\partial x} + v \frac{\partial g}{\partial y} + \ldots \]

so \[ \dot{u} = u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} \quad \Rightarrow \quad \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \]

\[ A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \]

Jacobian \(A\) at the Fixed Point.

This is called the Jacobian!
Now we analyze $\dot{x} = Ax$ where $x = \begin{bmatrix} u \\ v \end{bmatrix}$ using linear methods. This will tell us about the fixed point.

**Note:** If this analysis results in a borderline case, such as a **star**, **center**, or **degenerate node**, we must be careful! The small "neglected" nonlinear terms may play a role.

**Example:** Analyze the nonlinear system

\[
\begin{align*}
\dot{x} &= -x + x^3 \\
\dot{y} &= -2y.
\end{align*}
\]

1. Find all possible fixed points...

\[-x + x^3 = 0 \quad x(-1 + x^2) = 0 \quad x = 0, x = -1, x = 1
\]
\[-2y = 0 \quad \Rightarrow \quad y = 0
\]

Three fixed points \((0,0), (-1,0), (1,0)\)

2. Calculate the Jacobian

\[
J = \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix} = \begin{bmatrix}
-1 + 3x^2 & 0 \\
0 & -2
\end{bmatrix}
\]

the Jacobian can have functions inside!

3. Now consider each fixed point individually...

**a.** FP: \((0,0)\)

\[
A = J_{(0,0)} = \begin{bmatrix}
-1 & 0 \\
0 & -2
\end{bmatrix}
\]

\[
\begin{align*}
\mu &= -3 \\
\Delta &= 2 \\
\lambda_1 &= -1 \\
\lambda_2 &= -2
\end{align*}
\]

\[
\lambda = -\frac{3 \pm \sqrt{9 - 8}}{2} = -\frac{3 \pm 1}{2}
\]

This is a **stable node**

b) FP: \((-1, 0)\)

\[
A = J = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}
\]

\[c = 0 \text{ and } \lambda = \pm \frac{\sqrt{0 + 16}}{2} = \pm 4\]

\[\lambda_1 = 2 \quad \lambda_2 = -2\]

This is a saddle point.

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c) FP: \((1, 0)\)

\[
A = J = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}
\]

same as \((-1, 0)\)

This is a saddle point.

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4. **FIND NULLCLINES FROM ORIGINAL SYSTEM**

\[
\begin{align*}
\dot{x} &= -x + x^3 = 0 & x &= 0 \\
\dot{x} &= x = 1 \\
\dot{y} &= -2y = 0 & y &= 0
\end{align*}
\]

---

5. **Phase Portrait.**

- **#1 Draw Your FP's**
- **#2 Draw the nullclines**
- **#3 Decide on direction for nullclines.**
  - Ex. For the \(y=0\) nullcline only have flow in the \(x\)-direction
  - so look at \(\dot{x} = -x + x^3\) around FP to look for pos or neg flow

- **#4 Look at what type of node it is -** From linearization analysis -
  - to fill in the details

**NOTE:** you can always check phase portrait computer plot to make sure this is correct.
QUICKLY... Existence and Uniqueness... generalized from the 1-D case.
Given \( \frac{dx}{dt} = f(x) \), \( x(0) = x_0 \).
Suppose \( f \) is continuous and that all of its first
derivatives are continuous \( \frac{\partial f_i}{\partial x_j} \), \( i,j = 1, \ldots, n \)
on some open connected set \( D \subset \mathbb{R}^n \). Then
for \( x_0 \in D \), the initial value problem has
a solution on some time interval \((-\varepsilon, \varepsilon)\) about \( t = 0 \)
and the solution is unique.

An important idea comes from this...

If a solution exists and is unique \( \rightarrow \) different trajectories
never intersect.

Phase Portraits - always look "well-groomed".

OTHER COOL IDEAS...

If we can find a closed orbit in our phase portrait
then any trajectory starting
inside must stay inside.

Further... if there are no fixed points
inside the closed orbit...
Intuition \( \Rightarrow \) the trajectory can't
go forever w/o crossing.

POINCARE-BENDIXSON THEOREM... If a trajectory is
confined to a closed, bounded region and
there are no fixed points in the region,
then the trajectory MUST approach a
closed orbit.

BREAK... plot phase portraits on pplane - start HW.
If we are interested in finding quantitative aspects - we would need to do numerical computation.

Always use Runge-Kutta in vector form:

\[
\begin{align*}
\tilde{k}_1 &= f(\tilde{x}_n) \Delta t \\
\tilde{k}_2 &= f(\tilde{x}_n + 0.5 \Delta t) \tilde{k}_1 \\
\tilde{k}_3 &= f(\tilde{x}_n + 0.5 \Delta t) \tilde{k}_2 \\
\tilde{k}_4 &= f(\tilde{x}_n + \Delta t) \tilde{k}_3 \\
\tilde{x}_{n+1} &= \tilde{x}_n + \frac{1}{6} (\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4)
\end{align*}
\]

Don't let the vectors fool you... this is pretty much the same as 1-D.

When in standard form for 2-D

\[
\begin{align*}
x_{n+1} &= x_n + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) \\
y_{n+1} &= y_n + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right)
\end{align*}
\]

Then to get the phase portrait from this we must run the code for lots of different initial conditions.

run 1 \( \tilde{x}(0) = [0] \) nothing happens...
run 2 \( \tilde{x}(0) = [5] \)
run 3 \( \tilde{x}(0) = [4] \)
run 4 \( \tilde{x}(0) = [-1] \)

After doing a bunch of these you will get a phase portrait.