

Wed. 9/25 Thurs 9/26 Fri., 9/27	3.1-.2 Laplace & Images <b>Review</b>	<i>Summer Science Research Poster Session: Hedco7pm~9pm</i>	HW3
Mon. 9/30	<b>Exam 1 (Ch 2)</b>		

## Materials

## Announcements

## Last Times

All of last chapter / this course so far has been about stuff you already knew once – stuff that we'd actually met in Phys 232, though you may have forgotten it, and we were a little more timid of the integrals then. The only exception was Poisson's Equation and, when there is no charge density, Laplace's equation. When we met those, they were billed as being extremely useful since they're often much easier to solve than any of our other relations between field or voltage and charge for any but the simplest charge distributions.

## This Time

Now we're going to get a little more familiar with these equations – the character of their solutions, the *uniqueness* of their solutions, and some methods of solving them (we'll just barely get at the latter today, and carry on with it Monday).

## Summary

### *Solutions to Laplace's Equation*

Poisson's equation:  $\nabla^2 V = -\rho/\epsilon_0$

In empty regions ( $\rho=0$ ):  $\nabla^2 V = 0$

$$\nabla^2 V = \vec{\nabla} \cdot (\vec{\nabla} V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

### In 1-D:

In this case, the single derivative is not a partial derivative:

$$\frac{d^2 V}{dx^2} = 0$$

This differential equation is easy to solve:

$$\frac{dV}{dx} = \text{constant} = m$$

Integrate the equation above

$$\int dV = \int m dx$$

$$V = mx + b$$

This is the equation of a straight line. Two features of this solution also carry over to the 2-D and 3-D cases:

- (1) There are no local maxima or minima of  $V$ .

This follows from (1) because the average cannot be greater than (or less than) all of the parts.

- (2) The value of the potential at a point  $V(x)$  is the average of the values at the same distance  $R$  from the point.

$$V(x) = \frac{1}{2} [V(x-R) + V(x+R)]$$

This is easy to see in 1-D because the slope is positive. This holds for any value of  $R$  as long as all of the points being averaged are in empty space (solutions to Laplace's equation).

Example: Large parallel plate capacitor ( $E$  is constant, so  $V$  is linear)

### In 2-D:

In this case, there is a partial differential equation, which is more difficult.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

- (1) There are no local maxima or minima of  $V$ . (Still follows from 1) Note that the constraint is loosened on any one dimension – at some point there can be a concavity in the  $x$ -direction, but there must be an equal and *opposite* concavity in the  $y$ -direction,

$$\frac{\partial^2 V}{\partial x^2} = - \frac{\partial^2 V}{\partial y^2}$$

describing a saddle point, *not a local max or min*.

For a marble on such a surface, there's always a way into a spot and a way out of a spot.

- (2) The value of the potential at a point  $V(x, y)$  is the average of the values at the same distance  $R$  from the point.

$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle of radius } R} V dl$$

This is not a line integral, but adding up all of the values on a circle. (This property is proved for 3-D in the text.)

Examples: Any long (infinite) “tube” that doesn't change shape in one direction.

**In 3-D:**

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Again,

- (1) There are no local maxima or minima of  $V$ . (Still follows from 1). Again, this means there's always a way in and a way out of a spot.
- (2) The value of the potential at a point  $V(x, y, z)$  is the average of the values at the same distance  $R$  from the point.

$$V_{ave} = \frac{\oint_{sphere} V(r) dA}{A_{sphere}}$$

**Problem 3.2.** In one sentence, justify **Earnshaw's Theorem**: A Charged particle cannot be held in a stable equilibrium by electrostatic forces alone.

Well, *right* where the charge is to be trapped, there are no charges, thus

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Which means the potential has *no local maxima or minima* – there's always a way out and a way in. Multiply that by the charge of particle to be trapped and it means there are no local maxima or minima in the potential energy, which means there's no stable equilibrium.

**Proof of point 1.** In the last chapter Griffith's looked at a problem with similar geometry; simply for the sake of being able to handle this kind of geometry yourself, it's worth looking carefully at how he does it. See Figure 3.3 Put the point of interest at the origin and the source charge some distance  $z'$  up the  $z$ -axis. Point 1 claims that the average potential on a spherical surface centered on the origin,  $V_{ave}$ , equals the potential at the origin,  $V_o$ .

$$V_o = \frac{1}{4\pi\epsilon_0} \frac{q}{z}$$

$$V_{ave} = \frac{\sum_{sphere} V(r) \Delta A}{A_{sphere.total}}$$

which is, we integrate the voltage over the whole surface and divide by the surface area

$$V_{ave} = \frac{\oint_{sphere} V(r) dA}{A_{sphere}} = \frac{\oint_{sphere} \left( \frac{1}{4\pi\epsilon_0} \frac{q}{|r|} \right) dA}{4\pi R^2}$$

Where

$$|\mathbf{r}| = \sqrt{(\mathbf{r} - \mathbf{z}')^2} = \sqrt{(\mathbf{r} - \mathbf{z}') \cdot (\mathbf{r} - \mathbf{z}')} = \sqrt{R^2 + z'^2 - 2Rz' \cos\theta}$$

Here, since the points where we're evaluating the voltage are all on the surface of a sphere of radius  $R$ , their position vectors all have the same magnitude:  $|\mathbf{r}| = R$ .

$$V_{ave} = \frac{\oint_{sphere} \left( \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{R^2 + z'^2 - 2Rz' \cos\theta}} \right) dA}{4\pi R^2}$$

Recall that a nice differential patch of a spherical area is

$$dA = R d\theta (R \sin\theta d\phi)$$

So

$$V_{ave} = \frac{\int_{\phi=0}^{\phi=2\pi} \left( \int_{\theta=0}^{\theta=\pi} \left( \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{R^2 + z'^2 - 2Rz' \cos\theta}} \right) R^2 \sin\theta d\theta \right) d\phi}{4\pi R^2}$$

$$V_{ave} = \frac{1}{4\pi\epsilon_0} \frac{q 2\pi R^2 \int_{\theta=0}^{\theta=\pi} \frac{\sin\theta d\theta}{\sqrt{R^2 + z'^2 - 2Rz' \cos\theta}}}{4\pi R^2}$$

$$V_{ave} = \frac{1}{4\pi\epsilon_0} \frac{q 2\pi R^2 \int_{\cos\theta=1}^{\cos\theta=-1} \frac{-d(\cos\theta)}{\sqrt{R^2 + z'^2 - 2Rz' \cos\theta}}}{4\pi R^2}$$

$$V_{ave} = \frac{1}{4\pi\epsilon_0} \frac{q 2\pi R^2 \int_{u=-1}^{u=1} \frac{du}{\sqrt{R^2 + z'^2 + 2Rz'u}}}{4\pi R^2} = \frac{1}{4\pi\epsilon_0} \frac{q 2\pi R^2 2 \frac{1}{2Rz'} \sqrt{R^2 + z'^2 + 2Rz'u} \Big|_{u=-1}^{u=1}}{4\pi R^2}$$

$$V_{ave} = \frac{1}{4\pi\epsilon_0} q \frac{1}{2Rz'} \left( \sqrt{R^2 + z'^2 + 2Rz'} - \sqrt{R^2 + z'^2 - 2Rz'} \right) = \frac{1}{4\pi\epsilon_0} q \frac{1}{2Rz'} \left( \sqrt{R + z'} - \sqrt{R - z'} \right)$$

$$V_{ave} = \frac{1}{4\pi\epsilon_0} q \frac{1}{2Rz'} \left( R + z' - \sqrt{R - z'} \right)$$

I pause before canceling off the sqrt and square on the second term to observe that what it yields is

$$\sqrt{R - z'} = |R - z'| = z' - R$$

since  $R < z'$  for the charge being *outside* the circle.

Thus

$$V_{ave} = \frac{1}{4\pi\epsilon_0} q \frac{1}{2Rz'} \left( \sqrt{R+z'} - \sqrt{R-z'} \right)$$

$$V_{ave} = \frac{1}{4\pi\epsilon_0} q \frac{1}{2Rz'} (R+z' - (R-z'))$$

$$V_{ave} = \frac{1}{4\pi\epsilon_0} q \frac{1}{2Rz'} 2R$$

$$V_{ave} = \frac{1}{4\pi\epsilon_0} q \frac{1}{z'}$$

Whew! That's exactly the same as for the voltage right at the middle of the sphere. So indeed

$$V_o = V_{ave}.$$

Now, we just found that the voltage measured at a location, due to a single point charge, is equal to the average of the voltages over a sphere encircling the point (in a space where there are no sources, i.e., where  $\nabla^2 V = 0$ ).

Using the superposition principle, we can argue that if we added a second source charge to the scenario, then the voltage at a point due to *it* would be equal to the average of the voltages around the point – so the voltage at a point *due to the pair* of sources would equal the average voltage on a sphere encircling it. So too for 3 sources, 4 sources, ... N sources... any conceivable distribution of source charges (as long as we consider a sphere in the space where  $\nabla^2 V = 0$ .)

**Exercise: Problem 3.3** If  $V$  depends only on  $r$ , find the general solution in spherical, and then cylindrical coordinates. (see the Laplacian in terms of cylindrical and spherical coordinates inside the front cover).

**Spherical:**

$$\begin{aligned}\nabla^2 V(r) &= 0 \\ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V(r)}{\partial r} \right) &= 0 \\ r^2 \frac{\partial V(r)}{\partial r} &\equiv \text{Const} \\ \frac{\partial V(r)}{\partial r} &\equiv \frac{C}{r^2} \\ \partial V(r) &= \frac{C}{r^2} \partial r \\ \int_{V(r_0)}^{V(r)} \partial V(r) &= \int_{r_0}^r \frac{C}{r^2} \partial r \\ V(r) - V(r_0) &= -\frac{C}{r} + \frac{C}{r_0} \\ V(r) &= -\frac{C}{r} + \frac{C}{r_0} + V(r_0) \\ V(r) &= -\frac{a}{r} + b\end{aligned}$$

**Uniqueness Theorems** – If you find a solution, it is the right one!

- (1) If the potential  $V$  is known on all of the boundaries, the solution to Laplace's equation inside is unique. If there is a charge density inside and it is known, then the solution to Poisson's equation inside is unique.
  - a. **Proof.** Assume false and disprove that assumption: in our region both  $V_1$  and  $V_2$  satisfy Poisson's equation and boundary conditions, that is  $\nabla^2 V_1 = \nabla^2 V_2 = -\frac{1}{\epsilon_0} \rho$ . Clearly  $V_3 \equiv V_1 - V_2$  satisfies Laplace's equation,  $\nabla^2 V_3 = 0$ . But, if both  $V_1$  and  $V_2$  have the same known value at each point on the boundary, then  $V_3 = V_1 - V_2 = 0$  at the boundary. But with no maxima or minima within the space, there's no way for  $V_1 - V_2$  to be anything but 0 *anywhere* in the space, i.e., they're the *same* solution.
- (2) In a volume surrounded by conductors and containing a specified charge density, the electric field is uniquely determined by that charge density and the *total charge* on each conductor.
  - a. This may sound somewhat surprising, but let's put it another way: *you* don't need to specify how the charges will distribute themselves on the conductor,

*nature dictates* the one, unique way they will distribute themselves, and thus the one unique field they will generate.

- b. **Proof.** This is mathematically trickier. Again *assumes* that there are two possible solutions and then proceeds to prove that they are universally equal to each other. It uses both integral and differential form of Gauss's Law, defines

$\vec{E}_3 \equiv \vec{E}_2 - \vec{E}_1$  as the difference between the two proposed solutions. It also

uses the product rule:  $\vec{\nabla} \cdot (\epsilon_0 \vec{E}_3) = \epsilon_0 \vec{\nabla} \cdot \vec{E}_3 + \vec{E}_3 \cdot \vec{\nabla} \epsilon_0$  but

$\vec{\nabla} \cdot \vec{E}_3 \equiv \vec{\nabla} \cdot \vec{E}_2 - \vec{\nabla} \cdot \vec{E}_1 = \frac{1}{\epsilon_0} \rho - \frac{1}{\epsilon_0} \rho = 0$  and of course  $\vec{\nabla} \epsilon_0 = 0$ , so

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E}_3) = 0 - E_3^2$$

So Integrating this over the volume enclosed by the conductor gives

$$\int_{vol} \vec{\nabla} \cdot (\epsilon_0 \vec{E}_3) d\tau = \int_{surf} (\epsilon_0 \vec{E}_3) \cdot d\vec{a}$$

Since the volume is enclosed by a conductor, i.e. an *equipotential* (since a conducting surface is an equipotential), we can pull the Voltage outside of the integral over the surface area.

$$\int_{vol} \vec{\nabla} \cdot (\epsilon_0 \vec{E}_3) d\tau = \int_{surf} (\epsilon_0 \vec{E}_3) \cdot d\vec{a} = V_3 \int_{surf} \vec{E}_3 \cdot d\vec{a} = V_3 \int_{surf} (\vec{E}_1 - \vec{E}_2) \cdot d\vec{a} = V_3 \left( \frac{Q_{encl}}{\epsilon_0} - \frac{Q_{encl}}{\epsilon_0} \right) = 0$$

$$\int_{vol} \vec{\nabla} \cdot (\epsilon_0 \vec{E}_3) d\tau = \int_{vol} E_3^2 d\tau$$

But the integrand in this form is clearly never negative, so the only way it can integrate to 0 is if it is always 0. That is  $\vec{E}_3 \equiv \vec{E}_2 - \vec{E}_1 = 0$  and there's only one possible field in the space enclosed by the conductor.

### Methods of Finding Solutions

We are going to discuss three methods of solving Laplace's equation:

- (1) *Method of Images* – replace a problem with a simpler equivalent one (based on corollary of the first uniqueness theorem)
- (2) *Relaxation Method* – a computational method based on the potential at a point being the average of the values at the same distance (more about Next Time).
- (3) *Multipole Expansion* – a method for getting approximate answers for  $V$  far from a charge distribution

### Method of Images.

**Images and Mirrors.** A conductor acts like a ‘mirror’ to a charge. In a fundamental way, this isn’t just *simile*, this *is* why a mirror (typically made of a thin layer of metal protected by glass) works. That is, if there’s a real source on one side, when you look at the conductor, its surface charge arranges itself to produce a field as if there were another charge somewhere on the other side. In the case of you standing in front of the mirror and looking at your own image – you are the source of the light (electric and magnetic fields) that impact the mirror (sure a light bulb shown on you, but in a fundamental way, the light that makes it to the mirror is largely what you generated in response to the bulb’s light hitting you.)

That said, the conditions that define an *optical* image aren’t the exact same as define a *charge* image. Now, when we figure out where a charge image is, we’re asking where do the new fields point back to, when we’re asking where an ‘optical’ image is, we’re asking where do the light waves trace back to (where, as we’ll learn much later, the electric field of a light wave is *perpendicular* to the direction of its propagation). So these two types of images aren’t necessarily in the same place, but some of the intuition you have about one kind of image applies to the other.

For example, only rotationally symmetric surfaces create *true* images. That is to say a plane creates an image, a sphere creates an image (more or less), but a cylinder *doesn’t* create an image (of a point).

The exact condition that we’re using for placing a charge image is that, since the conductor’s surface defines an equipotential,

$$\frac{q_o}{r_o} + \frac{q_i}{r_i} = \text{const for all points on the surface.}$$

(note: if you have a collection of ‘object’ sources and ‘image’ sources, then the condition is that the sum over terms like these for all of them comes to 0.)

**We’ll really get our hands dirty with images after the test.**



"To what degree will we be solving Laplace's equation directly, i.e. solving using PDE solving methods?"

[Davies](#) [Hide response](#) [Post a response](#)

[Admin](#)

It looks like the next section goes into separation of variables at least briefly.

[Ben Kid](#)

"How much math in this book will be the same as PDE? The math is about to be the same in both classes. Will this continue, or will we quickly move on into math from other courses? Will this class reach parts of PDE before we get to them this semester?"

[Anton](#) [Post a response](#)

[Admin](#)

"I was a little confused with the introduction of the corollary. Does this mean that the first uniqueness theorem is only applicable to volumes that do not contain any charge but do have  $V$  specified at the boundaries?"

[Ben Kid](#) [Post a response](#)

[Admin](#)

"I found this reading very challenging, so I'm hoping we can get more of a lecture to clarify the reading for us, rather than work on a lot of examples by ourselves on the white boards, since this seems to be a very important section of the course."

[Sam](#) [Post a response](#)

[Admin](#)

"I'm curious, and I know we haven't yet gotten to the punch line, but are we so interested in Laplace's equation because ultimately we want a way of modeling the fields between charged particles and distributions? Or is there some other reason?"

[Casey McGrath](#) [Hide response](#) [Post a response](#)

[Admin](#)

At first it seemed a bit odd to me that we weren't as concerned about Poisson's equation, when it seems like there is intrinsically more information contained in the equation.

[Casey McGrath](#)

"I would like to go over the uniqueness theorems, I don't think I fully grasp the concept depicted in the book."

[Antwain](#) [Post a response](#)

[Admin](#)

"Are there any tricks for picking charge distributions for the method of images?"

[Spencer](#) [Post a response](#)

[Admin](#)

"Can we talk about the example of Purcell's on pg 123-124 where he is trying to describe the second uniqueness theorem?"

[Jessica](#) [Post a response](#)

[Admin](#)

"I could probably attribute it to just seeing completely new material as opposed to reviewing up until now, but I was just wondering if we could spend a little extra time doing conceptual work tomorrow. He lost me in more than one place. Thanks!"

[Rachael Hach](#)