Physics 332: Int. E&M   2013   A little More *A*, Multipole Expansion

Mon., 10/28 Wed., 10/30 Thurs. 10/31 Fri., 11/1	**5.4.3** Multipole Expansion of the Vector Potential **7.1.1-7.1.3** *Ohm's Law & Emf* **7.1.3-7.2.2** *Emf & Induction*	HW7
Mon. 11/4	**Exam 2 (Ch 3 & 5)**	

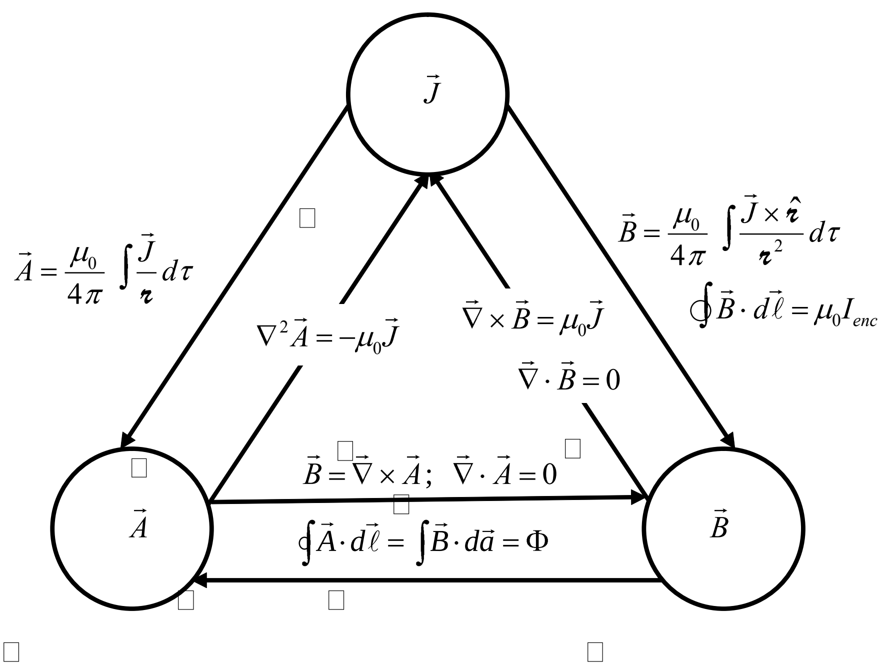
Announcements:

- Test in 1 week.
- Note re-ordering - moved Ch 10 after Ch 7 (ends up I need something from Ch 7 for Ch 10)

Last Time

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

<u>Summary of Magnetostatics</u> – This is similar to the diagram that we had for electrostatics, but all three quantities are vectors. Also, we haven't come up with a relation for one link.



Problem 5.25 (a) – Long, Thin Wire: Find the vector potential of a thin wire carrying current *I*.

Let's say that the current is in the +*z* direction. We know that the current will be in the same direction and by symmetry its size can only depend on the distance from the axis, so $\vec{A} = A(s)\hat{z}$ (in cylindrical coordinates). We also know that the magnetic field produced by the wire is $\vec{B} = (\mu_0 I/2\pi s)\hat{\phi}$. The definition of the magnetic vector potential, $\vec{B} = \vec{\nabla}\times\vec{A}$, gives us a differential equation:

$$\left(\frac{\mu_0 I}{2\pi s}\right)\hat{\phi} = -\left(\frac{\partial A}{\partial s}\right)\hat{\phi}.$$

Integrating this equation gives

$$A = -\int\left(\frac{\mu_0 I}{2\pi s}\right)ds = -\frac{\mu_0 I}{2\pi}(\ln s + C) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{s}{a}\right) \quad \text{and} \quad \vec{A} = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{s}{a}\right)\hat{z},$$

where $C = -\ln a$ to make the units look better.

Check the curl and divergence (in cylindrical coordinates). The only non-zero term in the curl is

$$\vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} = \frac{\mu_0 I}{2\pi s}\hat{\phi} = \vec{B}.$$

The divergence is

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = 0,$$

because A_z doesn't depend on *z*.

Problem 5.23: What current density would produce the vector potential $\vec{A} = k\,\hat{\phi}$ (where *k* is a constant) in cylindrical coordinates?

The associated magnetic field is

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_\phi)\hat{z} = \frac{1}{s}\frac{\partial}{\partial s}(ks)\hat{z} = \frac{k}{s}\hat{z}.$$

The current density can be found using Ampere's law (differential form):

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J},$$

$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \frac{1}{\mu_0}\left(-\frac{\partial B_z}{\partial s}\right)\hat{\phi} = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\phi} = \frac{k}{\mu_0 s^2}\hat{\phi}.$$

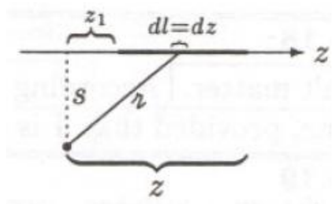
It circles the *z* axis like the vector potential. It would be difficult to use $\nabla^2\vec{A} = -\mu_0\vec{J}$ directly because the unit vectors have derivatives in cylindrical coordinates.

Problem 5.22: Find the vector potential for a current *I* along the *z* axis from z_1 to z_2.

Integrate up the contribution of each segment of the current using

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}d\tau' \rightarrow \frac{\mu_0}{4\pi}\int\frac{I\,d\vec{\ell}}{\mathscr{r}}.$$

If a point is a distance *s* from the *z* axis, then the separation is $\mathscr{r} = \sqrt{z^2 + s^2}$.



This gives (in cylindrical coordinates)

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2} \frac{dz}{\sqrt{z^2+s^2}}\,\hat{z} = \frac{\mu_0 I}{4\pi}\left[\ln\left(z+\sqrt{z^2+s^2}\right)\right]_{1}^{2}\hat{z},$$

$$= \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{z_2^2+s^2}}{z_1+\sqrt{z_1^2+s^2}}\right]\hat{z}.$$

<u>Boundary Conditions</u> – Suppose that there is a surface current on a boundary. How do the magnetic field and vector potential above and below the boundary compare?

Consider at a very thin pillbox that extends across the surface. Because the sides are very small, there is no magnetic flux through them. We know that $\oint \vec{B}\cdot d\vec{a} = 0$, so $B^{\perp}_{above} = B^{\perp}_{below}$ (they are in the same direction, too).

Consider an Amperian loop perpendicular to the surface current (as shown below). Applying Ampere’s law gives

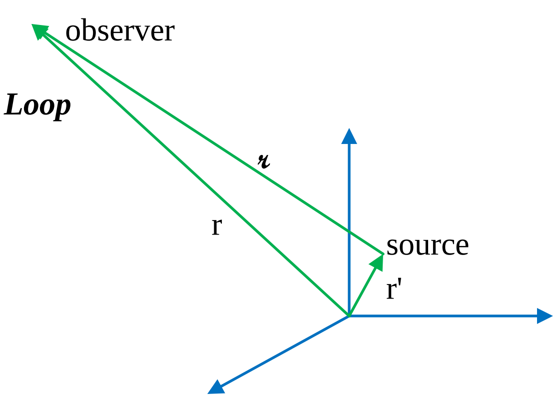
$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc},$$

$$B^{||}_{above}\ell - B^{||}_{below}\ell = \mu_0 K\ell,$$

$$B^{||}_{above} - B^{||}_{below} = \mu_0 K.$$

The components of the magnetic field parallel to the surface are perpendicular to the surface current.

The magnetic vector potential is continuous across the boundary, $\vec{A}_{above} = \vec{A}_{below}$, but it’s “derivative” isn’t because $\vec{\nabla}\times\vec{A} = \vec{B}$.



Multipole Expansion of the Vector Potential for a Current Loop

The vector potential for a current loop is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\oint \frac{\vec{I}(\vec{r}')}{\mathscr{r}}\,dl'$$

Where
$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}}$$

Now, it’s a mathematical truism that

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') = \frac{1}{r} + \frac{1}{r^2} r' \cos\theta' + \ldots$$

So, should we desire it, we could rewrite our integral for the vector potential as a sum of integrals

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\oint \vec{I}(\vec{r}')\,dl' + \frac{1}{r^2}\oint r' \cos\theta' \vec{I}(\vec{r}')\,dl' + \ldots\right]$$

Monopole Dipole

Why would we bother?

For the same reason that we bother writing a complicated mathematical function in terms of a Taylor series: so we can *approximate* the function when higher-order terms are small. In this case, that's when *r>>r'*.

One thing that's *nice* about these approximations is that they factor out the dependence on the source distribution (confined to the integrals) and the distance to the observer (the $1/r^n$ outside the integral.)

Warning – origin dependent: just like the values of the individual terms in a Taylor Series will be different depending upon the point around which you're expanding the function, the individual terms of the multipole expansion will be different depending upon where you put the origin. The closer you put it to the sources, the smaller *r'*s will be, and thus the better an approximation to keep only the first few terms in *r'/r*.

Now, let's return to our expansion and work with it a little.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\oint \vec{I}(\vec{r}')\,dl' + \frac{1}{r^2}\oint r' \cos\theta' \vec{I}(\vec{r}')\,dl' + \ldots\right]$$

If we have a steady current, then the magnitude of *I* can be factored out and we'll take the *dl* to be in the direction of the current flow.

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l}' + \frac{1}{r^2}\oint r' \cos\theta' d\vec{l}' + \ldots\right]$$

That first integral is then essentially the total displacement (say, of a charged particle) around the closed current loop; well that's 0! You're back where you started from.

$$\oint d\vec{l}' = 0$$

Thus there is *no* Monopole term.

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r^2}\oint r' \cos\theta' d\vec{l}' + \ldots\right]$$

So the Dipole term is the lowest-order survivor. You may recall that for the electric dipole term we rewrote

$$r'\cos\theta' = r'|\hat r|\cos\theta' = \vec r'\cdot\hat r$$

We'll do the same here

$$\vec A_{dipole}(\vec r) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint(\vec r'\cdot\hat r)\,d\vec l'$$

Now, Problem 1.62 walks one through the following identities

$$\oint(\vec c\cdot\vec r)\,d\vec l = \frac{1}{2}\left(\oint\vec r\times d\vec l\right)\times\vec c = \left(\int_S d\vec a\right)\times\vec c = \vec a\times\vec c$$

$$\vec A_{dipole}(\vec r) = \frac{\mu_0 I}{4\pi}\frac{\vec a\times\hat r}{r^2}$$

If we define the *dipole moment* as

$m \equiv I\int d\vec a'$ (this is often symbolized with μ.)

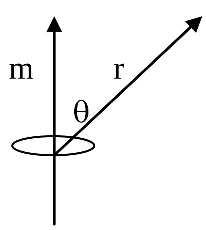
Then

$$\vec A_{dip}(\vec r) = \frac{\mu_0}{4\pi}\frac{\vec m\times\hat r}{r^2}$$

In the special case of a flat loop, the result can be written more simply, where $|\vec a|$ is the area and $\vec a$ points in the direction given by the RHR (fingers in direction of current, thumb in direction of magnetic dipole moment).

<u>Special Case</u>: If $\vec m$ is in the +z direction, then (see the diagram)

$$\vec A_{dip}(\vec r) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat\phi$$



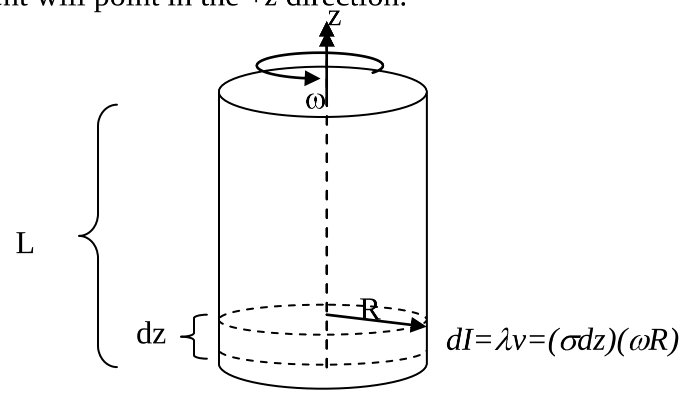
The magnetic field is found using $\vec B = \vec\nabla\times\vec A$. Of course, the results are only approximately true for large *r*. The magnetic field from a pure dipole and a physical dipole are very similar far away.

Now, we found this expression for the Dipole term of the Vector Potential only for a Current Loop. What if you have a surface current or a volume current? Build it up out of current loops.

<u>Examples/Exercises</u>:

Spinning Hollow Cylinder: Suppose a thin, hollow cylinder of radius *R* and length *L* carries a surface charge σ. If the cylinder rotates at angular frequency ω, what is its magnetic dipole moment?

Choose the axis of the cylinder to be the *z* axis and let the rotation be counterclockwise when viewed from the +*z* direction. By the RHR, the magnetic dipole moment will point in the +*z* direction.



Divide the cylinder into thin rings of height *dz*. The charge per length on one ring is $\lambda = \sigma dz$ and it is moving at a speed $v = \omega R$, so the current is $I = \lambda v = \sigma\omega R\, dz$. The area enclosed by the ring is πR^2, so the size of the magnetic dipole moment is

$$dm = dIa = \sigma\omega\pi R^3\ dz$$

Adding (integrating) up the contributions from all of the rings is easy because there is no *z* dependence. The total magnetic dipole moment (including direction) is

$$\vec{m} = \sigma\omega\pi R^3 L\ \hat{z}\ .$$

Now, what if it *were'nt* hollow? Now that we've got the moment for a single cylinder, we'll call *that* dm, and we'll demote R to r (variable of integration that will run from 0 to R.) Additionally, we'll say that the charge that we were attributing to a single patch of surface, now we'll say is for a tile of thickness dr.

$$d\vec{m} = \sigma\omega\pi r^3 L\ \hat{z}$$

$$\sigma = \rho dr$$

So

$$d\vec{m} = \rho dr\omega\pi r^3 L\ \hat{z}$$

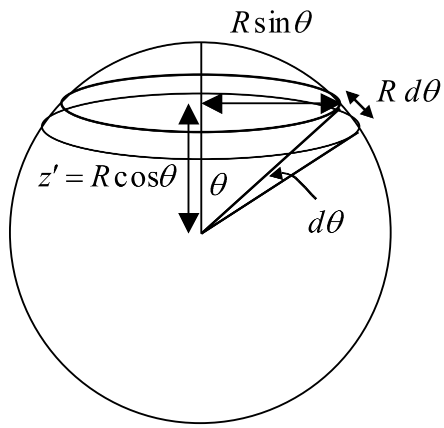
$$\vec{m} = \int_0^R \rho dr\omega\pi r^3 L\ \hat{z}$$

$$\vec{m} = \rho\omega\pi L\int_0^R r^3\ dr\hat{z} = \tfrac{1}{4}\rho\omega\pi L R^4\hat{z}$$

Spinning Hollow Sphere (compare with Example 5.11): Suppose a thin, hollow sphere of radius R carries a surface charge σ. If the cylinder rotates at angular frequency ω, what is its magnetic dipole moment? What is the approximate vector potential far from the sphere?

Choose the axis of the cylinder to be the z axis and let the rotation be counterclockwise when viewed from the $+z$ direction. By the RHR, the magnetic dipole moment will point in the $+z$ direction.

Consider a thin ring between the angles θ and $\theta + d\theta$ as shown below.



On the ring, $I = \lambda v$

The width of the ring is $R\,d\theta$, so the charge per length on it is

$$\lambda = \sigma R\,d\theta.$$

The radius of the ring is

$$s = R\sin\theta$$

, so points on it have a speed of

$$v = \omega R\sin\theta.$$

The current for the ring is

$$I = \lambda v = \sigma\omega R^2 \sin\theta\,d\theta.$$

The area enclosed by the ring is

$$a_{ring} = \pi\left(R\sin\theta\right)^2 = \pi R^2 \sin^2\theta$$

, so the size of its magnetic dipole moment is

$$dm = Ia = \pi\sigma\omega R^4 \sin^3\theta\,d\theta.$$

Adding (integrating) up the contributions from all rings gives

$$m = \pi\sigma\omega R^4 \int_0^{\pi} \sin^3\theta\,d\theta.$$

Use the relation $\sin^2\theta = 1 - \cos^2\theta$ to rewrite this as

$$m = \pi\sigma\omega R^4 \int_0^{\pi} \left(1 - \cos^2\theta\right)\sin\theta\, d\theta.$$

Make the substitution $u = \cos\theta$ and $du = -\sin\theta\, d\theta$ to get (don't forget the limits!)

$$m = -\pi\sigma\omega R^4 \int_1^{-1} \left(1 - u^2\right) du = -\pi\sigma\omega R^4 \left[u - \frac{u^3}{3} \right]_1^{-1} = \frac{4\pi\sigma\omega R^4}{3}.$$

Since the magnetic dipole moment is in the +z direction, the dipole term in the expansion of the vector potential is

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\phi} = \frac{\mu_0 \sigma\omega R^4 \sin\theta}{3r^2} \hat{\phi}$$

This happens to be the exact answer (see Example 5.11), but there is no way to know that in advance! We are only guaranteed that this approximation is valid for large *r*.

Note on Dipole Terms. It'll be a little while, but there *will* be a pay off for getting familiar with both electric and magnetic dipoles. When we loop back to Ch's 4 & 6 we'll talk about electric and magnetic fields in matter. These expressions provide useful tools for approximating how chunks of matter respond to fields.

"I don't completely understand why we end up integrating over the area of the current loop rather than doing a line integral to get the magnetic dipole moment. Also just out of curiosity what would you do to get the dipole moment of a non-flat loop?"
Ben Kid
"If we have time can we derive equation 5.84 which he invokes in the derivation of A_dip?"
Casey McGrath

"Can we do an example problem like ex 5.12 except with a more complicated geometry?"
Jessica

"How is it that the magnetic dipole approximation is identical in structure to the electric dipole approximation, and yet the actual fields look different, like Griffiths says? Wouldn't that suggest that one of the approximations doesnt work so well?"
Freeman,

"Can we just generally explain what the magnetic dipole moment actually is?"
Davies

"In what configuration would the monopole and dipole terms vanish? (Griffiths says it is rare). Also, I'm somewhat confused about Griffiths' comment on a current distribution whose potential is only dipole."
Spencer

"I had a little bit of trouble understanding the pattern of the multipole expansion of the vector potential."
Connor W,

"I do not think I am successfully corresponding figure 5.55 to the math discussed in this section. Is figure a. illustrating what the dipole would look like if the entire multipole expansion were carried out and figure b., the approximation? Thanks."
Rachael Hach

"I still don't fully understand what vector potential A is. I understand how it relates to other variables and how to solve for it but I still don't fully understand what it is we are trying to solve for in the physical sense. Could we go over this?"
Anton

"I still don't understand why the magnetic field outside of a solenoid is zero. I can't remember learning it in intro to E&M. Can someone clarify this for me?"
Casey P,

"I was confused by the derivation of the magnetic dipole moment esp. the step from equation 5.84 down to 5.86. Can we go over that?"
Sam