

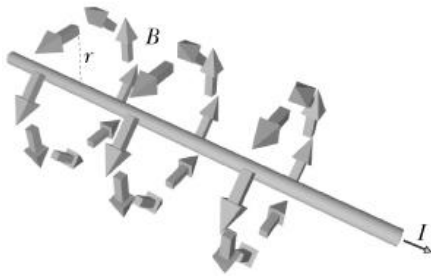
Mon. 10/21	(C 21.6-7,.9) 1.3.4-1.3.5, 1.5.2-1.5.3, 5.3.1-3.2 Div & Curl B	HW6
Wed. 10/23	(C 21.6-7,.9) 5.3.3-3.4 Applications of Ampere's Law	
Thurs. 10/24		
Fri. 10/25	1.6, 5.4.1-4.2 Magnetic Vector Potential	

### From the Past

#### *Biot-Savart Law*

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{q' \vec{v} \times \hat{r}}{r^2}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{d\vec{\tau}' \vec{J} \times \hat{r}}{r^2}$$



$$\vec{B} = \frac{\mu_0 I}{4\pi s} (\sin \theta_2 - \sin \theta_1) \hat{\phi} \quad \text{infinitely long: } \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

We're about to go down a very similar path with the magnetic field as we had at this point with the electric field – considering its divergence and curl. Since our basic program will be the same, while the math will be a little trickier, maybe it's worth refreshing our memories of how things played out with the electric field before embarking.

#### **Recall Gauss's Law for Electric Fields**

Back with the electric interaction, we started by meeting Coulomb's law; we didn't derive it, we just took it as a plausible given. So we very quickly defined the electric field:

$$\vec{E}_q(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

So, of course, if we had a distribution of charges, the field would be

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r^2} \hat{r} d\tau'$$

Given the geometry of that field, we were able to apply a few theorems from vector calculus to rephrase the relationship. For example, using the concept of *flux*, we reasoned out Gauss's (mathematical) Theorem and applied it to the electric field to get Gauss's (physics) Law.

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0} \text{ where it's only common sense that } Q_{encl} = \int \rho d\tau'.$$

We actually *proved* this to be the case for point charges, and thus for any combination thereof.

Then we made an interesting argument about what this relationship between the flux of electric field and enclosed charge implied for the *density* of charge and *density* of flux, a.k.a., divergence. We arrived at

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_{encl}}{\epsilon_0}$$

Really, in the great realm of mathematics, this relationship between the divergence of a vector field and the density of sources isn't unique to the electric field, it's quite general.

We also found that

$$\oint \vec{E} \cdot d\vec{l} = 0 \text{ went hand in hand with } \vec{\nabla} \times \vec{E} = 0$$

Far from being a not-so-interesting 0, this allowed us to define a scalar field whose (negative) gradient was the electric field, i.e. to define voltage.

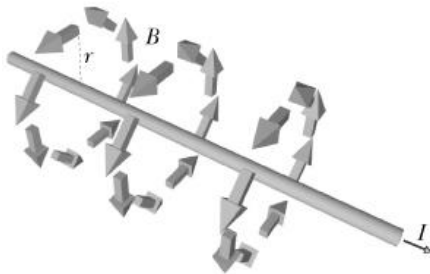
### Today: Div and Curl of B.

The book starts with the simple, 232-ish argument – considering an infinite line source and saying that most any more complicated current distribution could be built of a superposition of infinite line sources, so what holds for one holds for all.

Then it moves on to the more challenging general case.

Both treatments have their virtues – the infinite line source is *simple* and it's easy to visualize what's going on. The general case is more complex, but it familiarizes you with some more powerful reasoning – later on we're going to be looking at a still more complex situation (*non* steady currents), so it's worth our practicing on this, comparatively, simpler situation.

### Simple Approach – infinite Line current.



infinitely long:  $\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$  (expressed in cylindrical coordinates)

### Divergence of B

- **Conceptually:** divergence measures how much of an outward or inward flow there is / how much the vectors point *out* or *in*. This field *doesn't* point out or in at all, it points around and around, so

$$\circ \quad \vec{\nabla} \cdot \vec{B} = 0$$

- **Mathematically:**

$$\circ \quad \vec{\nabla} \cdot \vec{B}_{line}(\vec{r}) = \vec{\nabla} \cdot \left( \frac{\mu_0 I}{2\pi s} \hat{\phi} \right) = \frac{1}{s} \frac{\partial}{\partial s} \left( B_{line,s} \right) + \frac{1}{s} \frac{\partial}{\partial \phi} \left( B_{line,\phi} \right) + \frac{\partial}{\partial z} \left( B_{line,z} \right)$$

$$\circ \quad \vec{\nabla} \cdot \vec{B}_{line}(\vec{r}) = \frac{1}{s} \frac{\partial}{\partial s} (0) + \frac{1}{s} \frac{\partial}{\partial \phi} \left( \frac{\mu_0 I}{2\pi s} \right) + \frac{\partial}{\partial z} (0) = 0$$

- Again, by the superposition principle, this should hold for any current arrangement that can be built up out of *infinite, continuous* line currents.

### Curl of B

- **Conceptually:** Curl measures circulation, or how much the vector points *around*. Well, that's exactly what the magnetic field of an infinite line does. We can guess that, since the strength of that circulating field depends upon the strength of the current generating it we should get a curl that scales with I, but beyond that...

- **Mathematically: (ask them to do the derivative part)**

$$\circ \quad \vec{\nabla} \times \vec{B}_{line}(\vec{r}) = \vec{\nabla} \times \left( \frac{\mu_0 I}{2\pi s} \hat{\phi} \right) = -\frac{\partial}{\partial z} \left( B_{line,\phi} \right) \hat{s} + \frac{1}{s} \frac{\partial}{\partial s} \left( B_{line,\phi} \right) \hat{z}$$

$$\circ \quad \vec{\nabla} \times \vec{B}_{line}(\vec{r}) = \vec{\nabla} \times \left( \frac{\mu_0 I}{2\pi s} \hat{\phi} \right) = -\frac{\partial}{\partial z} (0) \hat{s} + \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\mu_0 I}{2\pi s} \right) \hat{z} = \frac{1}{s} \frac{\partial}{\partial s} \left( \frac{\mu_0 I}{2\pi} \right) \hat{z} = 0!$$

- What's going on here?

- **Singularity at s = 0.**

- We're imagining that all the current lives on an a line of *no thickness*. And so our representation of the magnetic field at s = 0 is a little hard to handle.
- If you haven't, read section 1.5 about the Dirac Delta Function. There you'll observe that,

$$\text{apparently } \vec{\nabla} \cdot \left( \frac{1}{r^2} \hat{r} \right) = 0!, \text{ but then again,..not quite. That}$$

section elaborates on the work-around for this case:

$$\vec{\nabla} \cdot \left( \frac{1}{r^2} \hat{r} \right) = 4\pi \delta^3(0); \text{ Similarly, } \vec{\nabla} \times \left( \frac{1}{s} \hat{\phi} \right) = 2\pi \delta^2(0)$$

- Work around: Stoke's Theorem

$$\int_{\text{surface}} \nabla \times \vec{v} \cdot d\vec{a} = \oint_{\text{path}} \vec{v} \cdot d\vec{l}$$

- We actually nearly proved this theorem back when we were considering the curl of the electric field; so now we'll just use it.

- Applying it to the magnetic field:

$$\int_{\text{surface}} \nabla \times \vec{B}_{\text{line}} \cdot d\vec{a} = \oint_{\text{path}} \vec{B}_{\text{line}} \cdot d\vec{l}$$

- Imagine a surface like a soap bubble through which the current passes, this surface is bound by a closed loop. We'll focus on integrating around that loop.
- Judging from the rotational symmetry of the magnetic field, it's tempting, but not necessary to use a circular loop that's centered on the current. It's not necessary since only the component of  $d\vec{l}$  that's parallel to the field (angular) survives the dot-product.

$$\oint_{\text{path}} \vec{B}_{\text{line}} \cdot d\vec{l} = \oint_{\text{path}} \frac{\mu_0 I}{2\pi s} \hat{\phi} \cdot (s d\vec{\phi} + d\vec{z}) = \oint_{\text{path}} \frac{\mu_0 I}{2\pi s} s d\phi$$

$$\oint_{\text{path}} \vec{B}_{\text{line}} \cdot d\vec{l} = \oint_{\text{path}} \frac{\mu_0 I}{2\pi} d\phi = \frac{\mu_0 I}{2\pi} \oint_{\text{path}} d\phi = \mu_0 I$$

- Now, rather than trying to unwrap the right-hand integral directly, the book takes the round-about approach by noting that

$$I_{\text{encl}} = \oint \vec{J} \cdot d\vec{a}'$$

$$\text{Since } \vec{J} = \frac{d\vec{l}}{da_{\perp}}$$

- Well, if we choose to add up all the current flowing through our soap bubble, then the areas are the same (sure, the current density is probably 0 most of the place, but that doesn't change the formal equality.)

$$\int_{\text{surface}} \nabla \times \vec{B}_{\text{line}} \cdot d\vec{a} = \mu_0 I = \mu_0 \int_{\text{surface}} \vec{J} \cdot d\vec{a}$$

- Apparently, the integrands are equal

$$\nabla \times \vec{B}_{\text{line}} = \mu_0 \vec{J}$$

- Again, this result should hold for any field that can be produced by a combination of *infinite and continuous* line charges.

### More Generally:

Now, we were rather restrictive in saying that this only applies for current distributions that can be built of infinite and continuous currents. What about a current loop? We could go through the math and demonstrate that it holds for that too, but then we could come up with an infinite number of other *what if* cases to explore. So we'll handle this completely generally. The result is that we'll be able to remove the *infinite* constraint, but we won't be removing the *continuous* constraint. In fact, you *don't* get this result for non-constant currents. (More on that much later.)

You'll notice that we ended up coming up with a relationship between B and J. So a good place for us to start is with our other relationship between the two. The Biot-Savart Law.

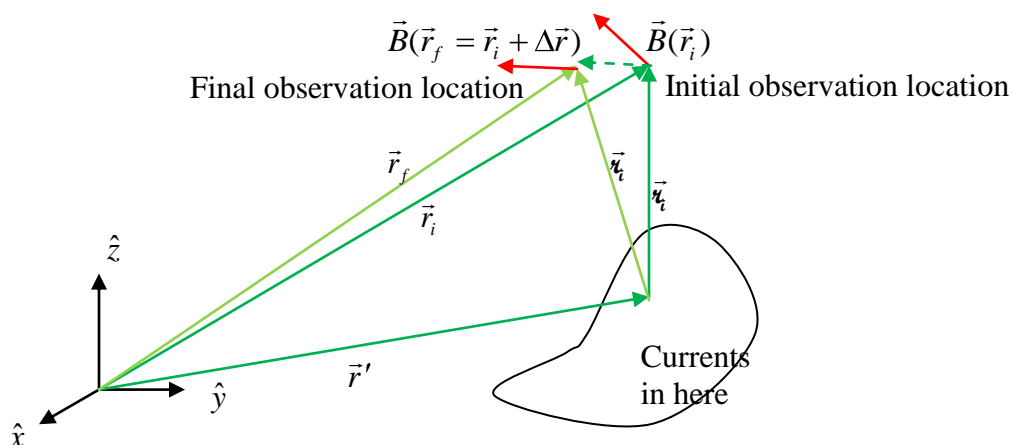
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \hat{r}}{r^2} d\tau' = \frac{\mu_0}{4\pi} \int \vec{J} \times \frac{\hat{r}}{r^2} d\tau'$$

### Divergence

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = \vec{\nabla} \cdot \left( \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \times \frac{\hat{r}}{r^2} d\tau' \right)$$

- **Variable of differentiation.**

- Before we distribute through the integral, it's worth pausing and thinking about exactly what it's taking the derivative of – the field as a function of *the observation location, r*. It's *not* taking the derivative of the current density as a function of source locations, *r'*. You might be thinking 'that's easy to say, but what the heck does it mean.' So let's look more carefully at what we mean by taking the derivative with respect to *r* but not *r'*.
- When we take the divergence of B, what we're asking is 'for a given source configuration, how does the field vary from one observation location to another?'
- Pictorially,



- Conceptually, when I find the curl or divergence of the field, I'm asking how the *field* differs at two different locations, not how the *source* differs at two different locations. In fact, we could, sum over all source's contributions to the field at one observation location, and then do the same at the other observation location, and compare the two. For example, if we wanted

- $\frac{d\vec{B}}{d\vec{r}} = \lim_{\Delta\vec{r} \rightarrow \infty} \frac{\vec{B}(\vec{r} + \Delta\vec{r}) - \vec{B}(\vec{r})}{\Delta\vec{r}}$  (where I'm using this cheaty notation of taking the derivative with respect to a vector to mean the divergence, but I want to keep this easy to look at)

- In terms of our integral, that's

$$\begin{aligned} \frac{d\vec{B}}{d\vec{r}} &= \lim_{\Delta\vec{r} \rightarrow \infty} \frac{\frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \times \frac{\vec{r} + \Delta\vec{r} - \vec{r}'}{|\vec{r} + \Delta\vec{r} - \vec{r}'|^3} d\tau' - \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} d\tau'}{\Delta\vec{r}} \\ &= \lim_{\Delta\vec{r} \rightarrow \infty} \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \times \left( \frac{\left( \frac{\vec{r} + \Delta\vec{r} - \vec{r}'}{|\vec{r} + \Delta\vec{r} - \vec{r}'|^3} - \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right)}{\Delta\vec{r}} \right) d\tau' \\ &= \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \times \left( \lim_{\Delta\vec{r} \rightarrow \infty} \frac{\left( \frac{\vec{r} + \Delta\vec{r} - \vec{r}'}{|\vec{r} + \Delta\vec{r} - \vec{r}'|^3} - \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right)}{\Delta\vec{r}} \right) d\tau' \\ &= \frac{d\vec{B}}{d\vec{r}} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \times \left( \lim_{\Delta\vec{r} \rightarrow \infty} \frac{\Delta}{\Delta\vec{r}} \left( \frac{\hat{z}}{r^3} \right) \right) d\tau' \end{aligned}$$

- Now, it's not *exactly* that easy, but the general idea is right – when we compare the field at two different observation locations, thus taking the derivative with respect to  $r$ , we're leaving the sources alone, thus not touching  $r'$ .

Okay back to the task at hand

$$\vec{\nabla}_{\vec{r}} \cdot \vec{B}(\vec{r}) = \vec{\nabla}_{\vec{r}} \cdot \left( \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \times \frac{\hat{z}}{r^2} d\tau' \right) = \frac{\mu_0}{4\pi} \int \vec{\nabla}_{\vec{r}} \cdot \left( \vec{J}(\vec{r}') \times \frac{\hat{z}}{r^2} \right) d\tau'$$

We could slip the del inside the integral since the integral sums over all the sources – that's not we're changing with the del, we're changing the *observation location*.

Just focusing on the integrand, we can apply the Product Rule 5 (inside front cover).

$$\vec{\nabla}_{\vec{r}} \cdot \left( \vec{J}(\vec{r}') \times \frac{\hat{z}}{r^2} \right) = \frac{\hat{z}}{r^2} \cdot \left( \vec{\nabla}_{\vec{r}} \times \vec{J}(\vec{r}') \right) - \vec{J}(\vec{r}') \cdot \left( \vec{\nabla}_{\vec{r}} \times \frac{\hat{z}}{r^2} \right)$$

Now, the first term is 0 simply because we're taking the derivative with respect to  $r$ , observation location, while the current density doesn't care where the observation location is, it depends on  $r'$ , where the charges are.

$$\vec{\nabla}_{\vec{r}} \cdot \left( \vec{J}(\vec{r}') \times \frac{\hat{z}}{r^2} \right) = 0 - \vec{J}(\vec{r}') \cdot \left( \vec{\nabla}_{\vec{r}} \times \frac{\hat{z}}{r^2} \right)$$

As for the remaining term, we can write it out in spherical coordinates (but we'll only bother with the  $r$ -hat dependent terms)

$$\vec{\nabla}_{\vec{r}} \times \frac{\hat{z}}{r^2} = \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left( \frac{1}{r^2} \right) \right] \hat{\theta} - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r^2} \right) \hat{\phi} = 0$$

So, as long as the Biot-Savart Law holds (which is as long as we have constant currents)

$$\vec{\nabla}_{\vec{r}} \cdot \vec{B} = 0$$

We have *derived* Gauss's Law for Magnetism from the Biot-Savart Law (in the case of constant currents.)

## Curl of B

That was the comparatively simple one to handle. Now it's time for the curl. We'll see if we can twist, bend, and otherwise contort it to find that a simple relationship between the curl of B and J follows from this relationship – is *derivable* from the Biot-Savart Law.

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left( \frac{\mu_0}{4\pi} \int \vec{J} \times \frac{\hat{z}}{r^2} d\tau' \right)$$

Again, before we distribute through the integral, it's worth pausing and recalling exactly what it's taking the derivative of – the field as a function of *the observation location*,  $r$ . It's *not* taking the derivative of the current density as a function of source locations,  $r'$ .

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left( \frac{\vec{J} \times \hat{z}}{r^2} \right) d\tau'$$

Now, let's just pull out, and focus on the integrand for a while.

$$\vec{\nabla} \times \left( \frac{\vec{J}(\vec{r}') \times \hat{z}}{r^2} \right) = \vec{\nabla} \times \left( \vec{J}(\vec{r}') \times \left( \frac{\hat{z}}{r^2} \right) \right)$$

(making explicit that J depends on  $r'$ , not  $r$ .)

Which product rule (inside front cover) applies here? #8

$$\begin{aligned} \vec{\nabla} \times \left( \vec{J} \times \left( \frac{\hat{z}}{r^2} \right) \right) &= \left( \left( \frac{\hat{z}}{r^2} \right) \cdot \vec{\nabla} \right) \vec{J} - \left( \vec{\nabla} \cdot \left( \frac{\hat{z}}{r^2} \right) \right) \vec{J} + \vec{J} \left( \vec{\nabla} \cdot \left( \frac{\hat{z}}{r^2} \right) \right) - \left( \frac{\hat{z}}{r^2} \right) \left( \vec{\nabla} \cdot \vec{J} \right) \\ &= 0 - \left( \vec{\nabla} \cdot \left( \frac{\hat{z}}{r^2} \right) \right) \vec{J} + \vec{J} \left( \vec{\nabla} \cdot \left( \frac{\hat{z}}{r^2} \right) \right) + 0 \end{aligned}$$

Now, both terms are a little tricky. Let's look at the second one first.

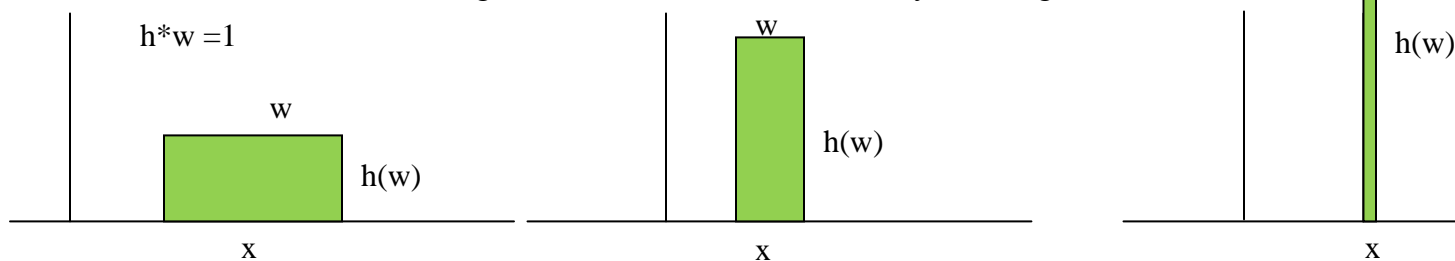
Just straight-forwardly taking this derivative will return 0, just like taking the curl of  $1/s$  did.

$$\vec{\nabla} \cdot \left( \frac{\hat{z}}{r^2} \right) = 0$$

$$\text{However, } \int \vec{\nabla} \cdot \left( \frac{\hat{z}}{r^2} \right) d\tau = \int \frac{\hat{z}}{r^2} \cdot d\vec{a} = \int \frac{1}{r^2} r^2 \sin\theta d\theta d\phi = 4\pi$$

So, it's a funny kind of 0, one that, when integrated over, isn't so 0 after all! We call that a dirac-delta. The idea is that, the function is 0 everywhere except at 0, where it's infinite in just the right way so that it sums to 1!

In 1-D it's not hard to imagine this as the limit of a curve, say a rectangle for whom



$$\delta(x) = \lim_{w \rightarrow 0} h(w) \text{ such that } h \cdot w = 1$$

You can imagine generalizing this to 2-D: the limit of the height of a cylinder as the radius goes to 0 and the volume is held equal to 1.

$$\delta^2(x, y) = \lim_{s \rightarrow 0} h(s)$$

For that matter, it can be generalized to 3-D (though harder to picture)

$$\delta^3(\vec{r})$$

Mathematically, we define a dirac delta function, symbolized  $\delta(\vec{r})$ , by .

$$\int \delta^3(\vec{r}) d\tau = 1$$

Returning to our conundrum, since  $\int \vec{\nabla} \cdot \left( \frac{\hat{z}}{r^2} \right) d\tau = 4\pi$ , we can say



$$\vec{\nabla} \cdot \left( \frac{\hat{\mathbf{z}}}{r^2} \right) = 4\pi\delta^3(\vec{r})$$

So can then write our integral as

$$\vec{\nabla} \times \vec{B} \stackrel{?}{=} \frac{\mu_0}{4\pi} \int -\hat{\mathbf{r}} \cdot \vec{\nabla} \left( \frac{\hat{\mathbf{z}}}{r^2} \right) + \vec{J} \left( \vec{\nabla} \cdot \left( \frac{\hat{\mathbf{z}}}{r^2} \right) \right) d\tau' = -\frac{\mu_0}{4\pi} \int \hat{\mathbf{r}} \cdot \vec{\nabla} \left( \frac{\hat{\mathbf{z}}}{r^2} \right) d\tau' + \frac{\mu_0}{4\pi} \int \hat{\mathbf{r}}(\vec{r}') 4\pi\delta(\vec{r}-\vec{r}') d\tau'$$

The effect the delta-function has is that, if you think of an integral as a sum, then all terms in that sum are 0 except where  $\vec{r} = \vec{0}$ , that is at  $\vec{r} - \vec{r}' = 0 \Rightarrow \vec{r}' = \vec{r}$ , that term just returns the integrand.

So, the second integral reduces to  $\mu_0 \vec{J}(\vec{r})$

$$\vec{\nabla} \times \vec{B} \stackrel{?}{=} -\frac{\mu_0}{4\pi} \int \hat{\mathbf{r}} \cdot \vec{\nabla} \left( \frac{\hat{\mathbf{z}}}{r^2} \right) d\tau' + \mu_0 \vec{J}(\vec{r})$$

Now, what about that *first* integral.

Griffith's pulls some nice moves to get it in terms of an area integral over J, and then he says that, as long as we *don't* have infinite currents, this term dies, leaving just

$$\vec{\nabla} \times \vec{B} \stackrel{?}{=} \mu_0 \vec{J}(\vec{r})$$

Note, we've independently shown that this holds when we *do* have infinite currents, and when we *don't*. So by superposition principle, it must hold even when we have a general mix of the two.

*Comparison of Magnetostatics and Electrostatics* (differential and integral forms)

Electrostatics – charges produces diverging electric fields

$$\text{Gauss's law:} \quad \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \quad \oint_S \vec{E} \cdot d\vec{a} = Q_{enc}/\epsilon_0$$

$$\text{(no name)} \quad \vec{\nabla} \times \vec{E} = 0 \quad \oint \vec{E} \cdot d\vec{\ell} = 0$$

Magnetostatics – moving charges (currents) produce curling magnetic fields

$$\text{(no name)} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \oint_S \vec{B} \cdot d\vec{a} = 0$$

$$\text{Ampere's law:} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

Both types of fields obey the superposition principle – add the fields produced by different sources (remember that they're vectors!).

In addition, we need the Lorentz force law,  $\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$ . Typically, electric forces are much larger than magnetic forces.

## **Preview**

For Wednesday, you'll review how to apply Ampere's law to find magnetic fields.

"Can we talk conceptually about the divergence and the curl of B because I don't understand what it is?"

[Jessica](#)

"The book says that the Dirac function when integrated from negative infinity to infinity is one. How do they derive this? Or is it just defined that way?"

[Casey P](#), AHoN swag 4 liphe

"Griffith's kind of skipped a step between 5.51 and 5.52 (he dropped two terms right away without writing them out). Could we write the full thing out like it should be so that we can see why the other two terms dropped off more easily?"

[Casey McGrath](#)

"Can we talk about why the divergence of B being zero is an important result and in what situations it will be useful to us?"

[Ben Kid](#) [Post a response](#)

[Admin](#)

"I was able to follow the concepts in the reading, except for the dirac delta function, especially the three dimensional case. So i was hoping we could talk about that a bit, Can we also see an example of actually doing the dirac integrals stepbystep?"

[Sam](#)

"Why do we want to find the curl of B?"

[Spencer](#)

"Griffiths refers to the divergence and integrations being in respect to the 'primed' and 'unprimed' variables. Could we go over what that means in respect to dot and cross products and del (div, curl) as well as integrals? This seems to be key here."

[Anton](#)

"On step 5.54 Why is the divergence of j 0?"

[Antwain](#)