Fri., 10/11	*(C 17)* **5.1.3** Lorentz Force Law: currents	
Mon. 10/14 Wed. 10/16 Thurs 10/17	**Study Day** *(C 17)* **5.2** Biot-Savart Law	HW5

From Last Time

For last time, you read a little about special relativity and how two objects' perspectives on things depend upon their relative velocity. Applying that to the interaction between charges – while a charge, who naturally thinks itself stationary, may think it's being acted upon by the good-old electric interaction (which only depends upon its charge), we, who see it moving, think there's also an aspect of the interaction that depends on that motion – magnetism.

Just running with that observation, we could, in very broad strokes say:

$$\vec{F} = Q\left[\vec{E} + \left(\vec{v} \times \vec{B}\right)\right]$$

This is almost definitional – it says that a charged particle can experience a force that only depends on its charge and a force that also depends on its velocity. This doesn't say much about what *causes* those forces, but it's not for no reason at all that we chose to express the second one in terms of a cross-product.

Why Cross-Product? We can get a hint at it's rational from noting that, according to Special Relativity, the dimension parallel to motion that's perceived to contract while the dimension perpendicular is not – as we've seen, that changes perceived charge densities off to the side of your motion, but not across your motion – and new charge densities at your sides means you get pushed sideways – so the force is *perpendicular to your motion* as the cross-product insures.

__Examples/Exercises__:

Problem 5.2 (a) & (c)

For both parts, the particle starts from the origin, $y(0) = 0$ and $z(0) = 0$, so we know that $C_3 = -C_1$ and $C_4 = -C_2$. Taking the time derivative of the general solution gives

$$\dot{y}(t) = -C_1\omega \sin \omega t + C_2\omega \cos \omega t + E/B,$$
$$\dot{z}(t) = -C_2\omega \sin \omega t - C_1\omega \cos \omega t.$$

a. We are also given $\vec{v}(0) = (E/B)\hat{y}$, so $\dot{y}(0) = E/B$ and $\dot{z}(0) = 0$. The first condition implies that $C_2 = 0$, which means that $C_4 = 0$. The second condition implies that $C_1 = 0$, so $C_3 = 0$.

   The solution that satisfies the initial conditions is

$$y(t) = (E/B)t,$$
$$z(t) = 0.$$

In other words, the particle moves with a constant velocity $\vec{v}(t) = (E/B)\hat{y}$. The electric and magnetic forces balance at this velocity (perpendicular to *E* and *B* at the right speed).

c. We are also given $\vec{v}(0) = (E/B)(\hat{y} + \hat{z})$, so $\dot{y}(0) = E/B$ and $\dot{z}(0) = E/B$. The first condition implies that $C_2 = 0$, which means that $C_4 = 0$. The second condition implies that $C_1 = -E/\omega B$, so $C_3 = E/\omega B$.

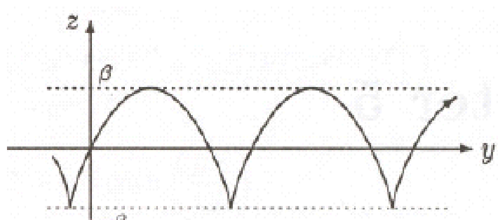
The solution that satisfies the initial conditions is

$$y(t) = -(E/\omega B)\cos\omega t + (E/B)t + (E/\omega B) = (E/\omega B)[1 + \omega t - \cos\omega t],$$
$$z(t) = (E/\omega B)\sin\omega t.$$

With just the cosine in *y*(*t*), this would be clockwise motion around a circle of radius $E/\omega B$ with a angular frequency *ω*. The particle is moving clockwise around a circle whose center is moving along the *y* axis. Another way to see this is to isolate the sine and cosine, square and add them, which gives one. That expression can be rearranged to give

$$\left[y(t) - \frac{E}{\omega B}(1 + \omega t)\right]^2 + [z(t)]^2 = \left(\frac{E}{\omega B}\right)^2.$$

This is the equation for a circle of radius $E/\omega B$ with the center at $y = E/\omega B(1 + \omega t)$ and *z* = 0. The graph is shown below.



(c)

Imagine charge's perspective. In light of the argument I presented last time for how a moving charge might perceive it's interaction with a current carrying wire, it's reasonable to ask 'how would a charge see this situation play out? First we need to get a little concrete about the source of the magnetic field. As you may recall from Phys 232, a solenoid, i.e., coil of wire, produces a fairly uniform field that runs parallel to its axis**.**

- **Frame Transformation with Solenoid**
  - **Lab Frame.** If you have a charged particle initially moving radially in a solenoid, then the magnetic interaction will push it sideways. In this reference frame, the current in the solenoid is circling with uniform speed and there's no net charge on the wires (okay, a little charge gradient, or there *wouldn't be* a current, but aside from that). To make things concrete, let's say the current is circulating counter-clockwise (generating an upwards

magnetic field), the charged particle is initially moving down, and so it gets pushed *left*.

- **Particle's Frame.** Transform to the particle's reference frame – from its perspective, it's got no speed (it's the solenoid that's moving) so why does *it* think it gets pushed sideways?
  - Well, from *it's* perspective, the charges on the right of it are moving up faster than the charges on the left of it are moving down, and it sees a heavier charge density of those fast moving charges on the left than of the slow moving ones on the right. So it gets pushed *left.*

Summary

Current (*I*)

As we noted last time, the magnetic force is pretty darn small (as are most relativistic effects), so we won't get very far talking about its effect on individual charges, it's more practical to talk about the cumulative effect on whole streams of charges – currents.

Say you have a whole bunch of charges moving, each with their own velocities, then the total magnetic force on them is

$$\vec{F}_{mag} = \sum q_i \vec{v}_i \times \vec{B} = \int \rho d\tau \vec{v} \times \vec{B} = \int \rho \vec{v} \times \vec{B} d\tau$$

Of course, if they happen to be confined to living in a surface, then this reduces to

$$\vec{F}_{mag} = \int \sigma \vec{v} \times \vec{B} dA$$

And if they happen to be confined to living on a line, then this reduces to

$$\vec{F}_{mag} = \int \lambda \vec{v} \times \vec{B} dl$$

So, as would be expected, the force depends upon the location and velocity of each morsel of charge.

Current.

This is sufficient, we've got everything defined that we need; however, it's convenient to speak in terms of current: the rate of charge flow. $\vec{I} = \frac{d\vec{q}}{dt}$ (it's a vector because it says not just how much, but also in what direction it's moving).

The units are: 1 C/s = 1 amp (A).

- **Linear.** If the charges are confined to move along a line, then $\vec{I} = \frac{d\vec{q}}{dt} = \frac{dq}{dx}\frac{d\vec{x}}{dt} = \lambda \vec{v}$
  - To measure that, you can imagine someone watching a point on the wire and counting up the charges as they flow by.
  - And so the force acting on *all* those charges is

- $\vec{F}_{mag} = \int \lambda \vec{v} \times \vec{B} dl = \int \vec{I} \times \vec{B} dl$

- Again, since it's confined to move down the line, the direction of the flow can't help but be along the path of integration, so you can rewrite this as
    - $\vec{F}_{mag} = \int \lambda \vec{v} \times \vec{B} dl = \int I d\vec{l} \times \vec{B}$
- **Warnings:**
    - **Order.** First, the order in a cross-product *matters*. So you need to cross the dl into the B, and not the other-way-around (assume directions for the two and do the right hand rule for dl×B and then for B×dl and see that they point in opposite directions)
    - **Direction.** Second, normally, *dl* now needs to have the direction of the current flow *not* the opposite direction, so if the current's flowing in the –*x* direction, dl is –*dx*, not *dx*.

- **Plainar.** If the charges are free to move in a surface, then the force is
    - $\vec{F}_{mag} = \int \sigma \vec{v} \times \vec{B} dA = \int \vec{K} \times \vec{B} dA$
    - We define the surface current density
        - $\vec{K} \equiv \sigma \vec{v}$
    - For the sake of tying things together, we can relate this back to the current by

      $$\vec{K} \equiv \frac{dq}{dA}\frac{d\vec{l}_{||}}{dt} = \frac{dq}{dA}\frac{d\vec{l}_{||}}{dt} = \frac{d\vec{q}}{d(l_{||} \cdot l_{\perp})}\frac{dl_{||}}{dt} = \frac{dl_{||}}{d(l_{||} \cdot l_{\perp})}\frac{d\vec{q}}{dt} = \frac{d}{dl_{\perp}}\vec{I}$$

      $$\vec{K} = \frac{d\vec{I}}{dl_{\perp}}$$

        - To measure *that,* you can imagine someone laying a line across the sheet, perpendicular to the flow, and counting the charges crossing the line.

- **Volume.** If the charges are free to move throughout a volume, then the force is
    - $\vec{F}_{mag} = \int \rho \vec{v} \times \vec{B} d\tau = \int \vec{J} \times \vec{B} d\tau$
    - We define the volume current density
        - $\vec{J} \equiv \rho \vec{v}$
    - For the sake of tying things together, we can similarly relate this back to the current and we get
        - $\vec{J} = \frac{d\vec{I}}{da_{\perp}}$

- To measure *that*, you can imagine some one holding up a hoolahoop perpendicular to the flow and counting the charges flowing through it.

You can translate between equations for charges and various current densities using

$$\sum(\ )q_i\vec{v}_i \sim \int_{\text{line}}(\ )\vec{I}\,d\ell \sim \int_{\text{surface}}(\ )\vec{K}\,da \sim \int_{\text{volume}}(\ )\vec{J}\,d\tau .$$

Continuity Equation

Since

$$\vec{J} = \frac{d\vec{I}}{da_{\perp}}$$

the current (charge per time) crossing a surface is

$$I = \int_S \vec{J}\cdot d\vec{a}$$

(here, the area *vector* points out of the surface)

For a closed surface, we can apply the divergence theorem to get

$$I = \oint_S \vec{J}\cdot d\vec{a} = \int_V \left(\vec{\nabla}\cdot\vec{J}\right)d\tau . \qquad (1)$$

If the charge is flowing out this will be positive (since J and a point in the same direction).

Then again,

$$I = \frac{dq}{dt} = \frac{d}{dt}\int_V \rho\, d\tau = \int_V \frac{d\rho}{dt}\,d\tau$$

Now we defined a positive current was for charge flowing *out*, but the enclosed charge density would then be *decreasing*, so apparently we need to fix the sign to be consistent with the previous relation.

$$I = \int_V \left(-\frac{d\rho}{dt}\right)d\tau$$

Now comparing the two

$$\int_V \left(\vec{\nabla}\cdot\vec{J}\right)d\tau = \int_V \left(-\frac{d\rho}{dt}\right)d\tau$$

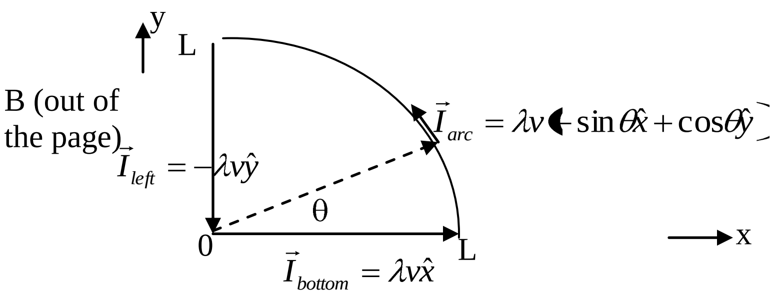
$$\vec{\nabla}\cdot\vec{J} = -\frac{d\rho}{dt}$$

This is known as the continuity equation since it says that, if you’re loosing charge somewhere, it’s got to be flowing.

In Ch. 5, we'll concentrate on "steady currents" which flow continuously for a long time with no charge piling up. (We learned in PHYS 232 that initially some surface charge does build up on wires.) For steady currents, there is no change in charge density anywhere, so $\vec{\nabla}\cdot\vec{J}=0$. This situation is known as *magnetostatics*, because it leads to constant magnetic fields.

Examples/Exercises:

Force on bent wire in uniform field:



$$\vec{F}_{mag} = \int dl\vec{I}\times\vec{B}$$

$$\vec{F}_{mag} = \int_{x=0}^{x=L} dl\vec{I}_b\times\vec{B} + \int_{\theta=0}^{\theta=\frac{\pi}{2}} dl\vec{I}_{arc}\times\vec{B} + \int_{y=0}^{y=L} dl\vec{I}_{arc}\times\vec{B}$$

$$\vec{F}_{mag} = \lambda vB\left(\int_{x=0}^{x=L} dx\hat{x}\times\hat{z} + \int_{\theta=0}^{\theta=\frac{\pi}{2}} Ld\theta(-\sin\theta\hat{x}+\cos\theta\hat{y})\times\hat{z} + \int_{y=0}^{y=L} dy(-\hat{y})\times\hat{z}\right)$$

$$\vec{F}_{mag} = \lambda vB\left(L(-\hat{y}) + \int_{\theta=0}^{\theta=\frac{\pi}{2}} Ld\theta(\sin\theta\hat{y}+\cos\theta\hat{x}) + L(-\hat{x})\right)$$

$$\vec{F}_{mag} = \lambda vB\left(L(-\hat{y}) + L(-\cos\theta\hat{y}+\sin\theta\hat{x})\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} + L(-\hat{x})\right)$$

$$\vec{F}_{mag} = \lambda vB\left(L(-\hat{y}) + L(\hat{y}+\hat{x}) + L(-\hat{x})\right) = 0$$

Alternatively, if B is in, say, the y direction,

$$\vec{F}_{mag} = \int dl\vec{I} \times \vec{B}$$

$$\vec{F}_{mag} = \int_{x=0}^{x=L} dl\vec{I}_b \times \vec{B} + \int_{\theta=0}^{\theta=\frac{\pi}{2}} dl\vec{I}_{arc} \times \vec{B} + \int_{y=0}^{y=L} dl\vec{I}_{arc} \times \vec{B}$$

$$\vec{F}_{mag} = \lambda v B\left(\int_{x=0}^{x=L} dx\hat{x} \times \hat{y} + \int_{\theta=0}^{\theta=\frac{\pi}{2}} Ld\theta\left(-\sin\theta\hat{x} + \cos\theta\hat{y}\right) \times \hat{y} + \int_{y=0}^{y=L} dy\left(-\hat{y}\right) \times \hat{y}\right)$$

$$\vec{F}_{mag} = \lambda v B\left(L(\hat{z}) + \int_{\theta=0}^{\theta=\frac{\pi}{2}} Ld\theta\left(-\sin\theta\hat{z}\right)\right)$$

$$\vec{F}_{mag} = \lambda v B\left(L(\hat{z}) + L(\cos\theta\hat{z})\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}\right)$$

$$\vec{F}_{mag} = \lambda v B\left(L(\hat{z}) + L(-\hat{z})\right) = 0$$

So, regardless of the direction of B, there's no *net* force on the loop. Mind you, there are different forces on different legs – making the loop want to rotate.

5.4 A square loop of side a lying in the yz plane and centered on the origin, carrying current I counterclockwise when viewed down the x axis. B = kz x-hat. What is the force?

5.5

Preview

For Wednesday, you'll review the Biot-Savart law for calculating the magnetic field due to a current.