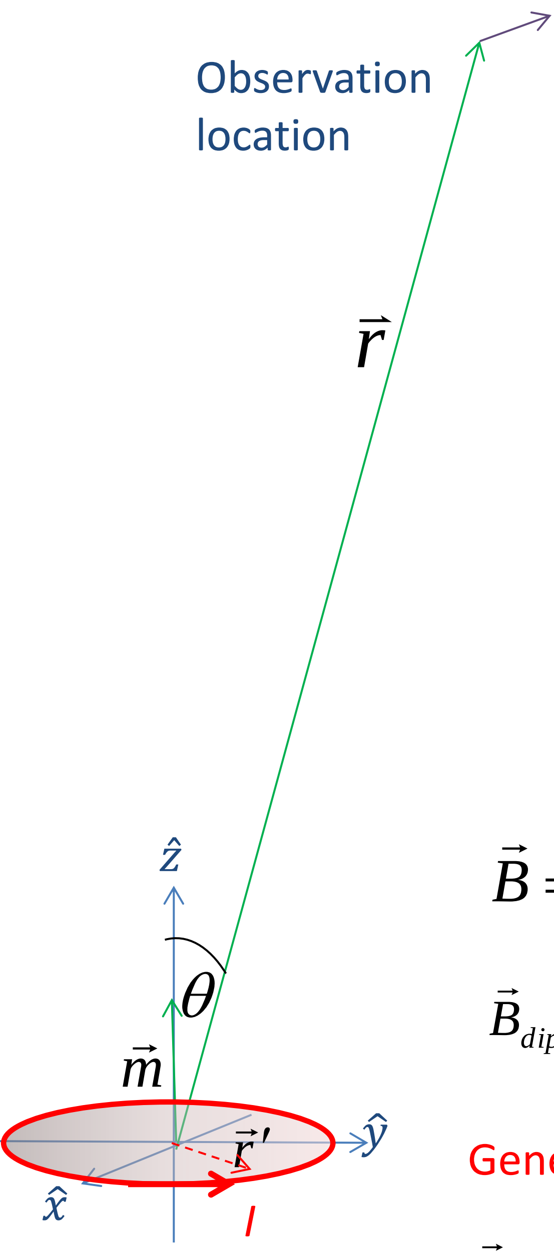
Fri.	6.2 Field of a Magnetized Object	
Mon. Tues.	6.3, 6.4 Auxiliary Field & Linear Media	HW9

Dipole term for a loop



$$\vec{A}(r) = \frac{\mu_o}{4\pi}\left(\frac{\vec{m}\times\hat{r}}{r^2} + \ldots\right)$$

Magnetic Dipole Moment

$$\vec{m} \equiv I\vec{a}'$$

$$\vec{m} = I\left(\pi R^2 \hat{z}\right)$$

$$\vec{A}(r) = \frac{\mu_o}{4\pi}\left(\frac{I\pi R^2 \hat{z}\times\hat{r}}{r^2} + \ldots\right) = \frac{\mu_o}{4\pi}\left(\frac{I\pi R^2 \sin\theta}{r^2}\hat{\phi} + \ldots\right)$$

Same direction as current

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{m}{r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$ (yes, same form as *E* for *p*)

Generally (*m*'s not necessarily at origin or pointing up)

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{3(\vec{m}\cdot\hat{\mathscr{r}})\hat{\mathscr{r}} - \vec{m}}{\mathscr{r}^3}$$

Magnetic Field Effects on Atomic Dipoles

Proto-Quantum Derivation

Consider a charged particle moving in the presence of a magnetic field.
The 'momentum' in the particle + field system: $\vec{p}_{system} = \vec{p}_{kin} + \vec{p}_{field} = m\vec{v}_1 + q\vec{A}$

$$\vec{p}_{kin} = \vec{p} - q\vec{A}$$

For an electron, $M = m_e$, $q = -e$

$$H^{*}_{Hamiltonian} = \frac{p_{kin}^2}{2M} = \frac{1}{2M}\left(\vec{p} - q\vec{A}\right)^2 = \frac{1}{2m_e}\left(\vec{p} + e\vec{A}\right)^2 = \frac{p^2}{2m_e} + \frac{e}{2m_e}\left(2\vec{A}\cdot\vec{p}\right) + \frac{e^2}{2m_e}A^2$$

Magnetic field uniform and in the z direction gives $\vec{A} = \frac{B_z s}{2}\hat{\phi} = \frac{\vec{B}\times\vec{r}}{2}$

$$H = \frac{p^2}{2m_e} + \frac{e}{2m_e}\left(\vec{B}\times\vec{r}\right)\cdot\vec{p} + \frac{e^2}{8m_e}\left(\vec{B}\times\vec{r}\right)^2$$

Product Rule 2

$$\vec{B}\cdot\left(\vec{r}\times\vec{p}\right)$$
$$\vec{B}\cdot\vec{L}$$

Product Rules 1 & 2 and define B in z direction, s in x-y plane, reduces to

$$\vec{B}\cdot\left(\vec{B}s^2\right) = \left(Bs\right)^2$$

$$H = \frac{p^2}{2m_e} + \left(\underbrace{\frac{e}{2m_e}\vec{L}\cdot\vec{B}}_{\text{Paramagnetic}} + \underbrace{\frac{e^2}{8m_e}\left(Bs\right)^2}_{\text{Diamagnetic}}\right)$$

Paramagnetic
Orbits orient to minimize energy-add to field

Diamagnetic
Opposes field

So if $m = -\frac{dH}{dB} = -\frac{e}{2m_e}L_z - \frac{e^2}{4m_e}Bs^2$

What kind of atom is predominately diamagnetic?

*not really how you build a Hamiltonian, but happens to work in this case

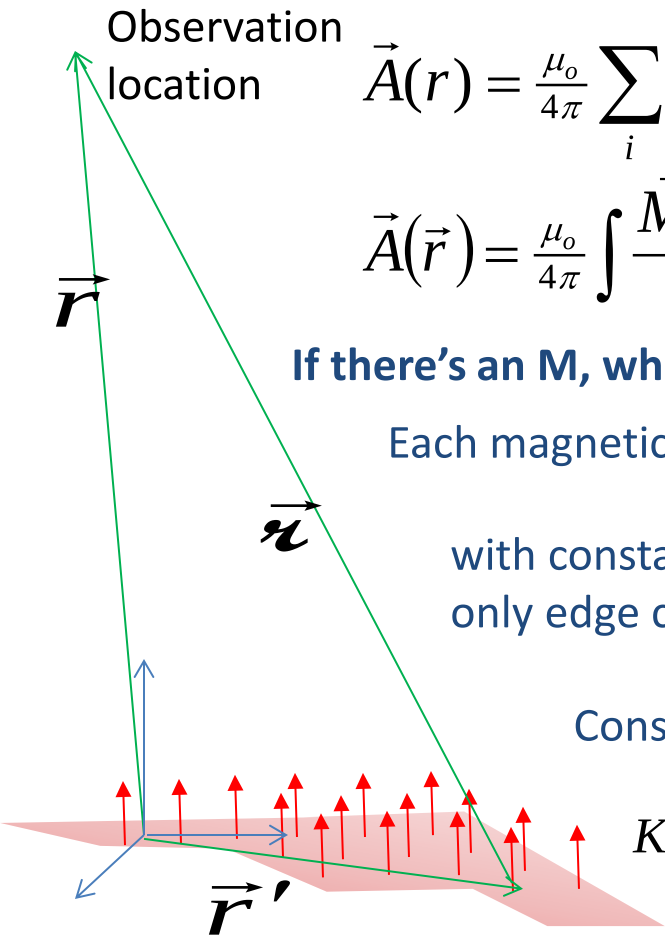
Magnetization

$$\vec{M} \equiv \frac{d\vec{m}}{d\tau'}$$

M≡ density of magnetic dipole moments

Consider patch of material covered in magnetic dipoles

If differentially small

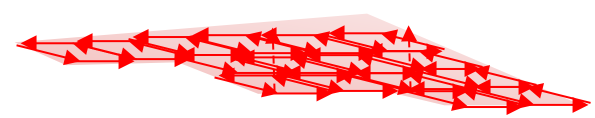


$$\vec{A}(r) = \frac{\mu_o}{4\pi}\sum_i \frac{\vec{m}_i \times \hat{\mathscr{r}}_i}{\mathscr{r}_i^{\,2}} = \frac{\mu_o}{4\pi}\int \frac{d\vec{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_o}{4\pi}\int \frac{d\tau'\left(\frac{d\vec{m}}{d\tau'}\right)\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

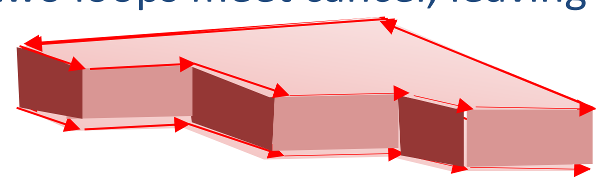
$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int \frac{\vec{M}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

If there's an M, what does that say about currents?

Each magnetic dipole is a current loop, so



with constant M, opposite currents where two loops meet cancel, leaving only edge current



Consider a block of some thickness

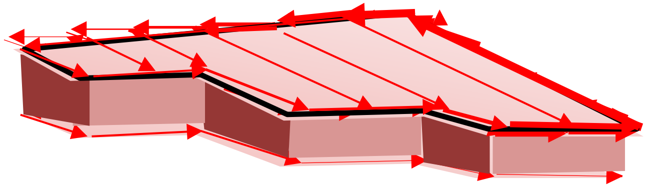
$$K_b = \frac{dI}{dl_{side}} = \frac{dIa_{\perp side}}{dl_{side}a_{\perp side}} \qquad \vec{K}_b = \frac{dI\vec{a}_{loop}\times\hat{n}}{d\tau} = \frac{d\vec{m}\times\hat{n}}{d\tau} = \vec{M}\times\hat{n}$$

Magnetization

$$\vec{M} \equiv \frac{d\vec{m}}{d\tau'}$$

Consider patch of material covered in differentially-small magnetic dipoles

If *not* equal magnetizations / currents, inner current crossings only *partially* cancel, giving current across body



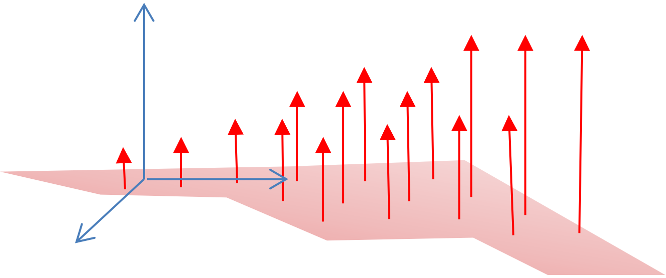
The current down one stripe is the difference between that around two adjacent loops: $I(y+\Delta y)-I(y)$

$$\vec{J} = \frac{d\vec{I}}{da_\perp} = \frac{I(y+\Delta y)-I(y)}{\Delta y \Delta z}\hat{x} = \frac{\frac{I(y+\Delta y)}{\Delta z} - \frac{I(y)}{\Delta z}}{\Delta y}\hat{x} = \frac{\frac{I(y+\Delta y)\Delta x \Delta y}{\Delta x \Delta y \Delta z} - \frac{I(y)\Delta x \Delta y}{\Delta x \Delta y \Delta z}}{\Delta y}\hat{x}$$

$$= \frac{\frac{I(y+\Delta y)a_{enc}}{\tau} - \frac{I(y)a_{enc}}{\tau}}{\Delta y}\hat{x} = \frac{M_z(y+\Delta y)-M_z(y)}{\Delta y}\hat{x} = \frac{\partial M_z}{\partial y}\hat{x}$$

$$\vec{J}_b = \vec{\nabla}\times\vec{M}$$

So, if the magnetization, which here point's z, varies across y, there's a net current in x Looks a lot like one term in a cross product.



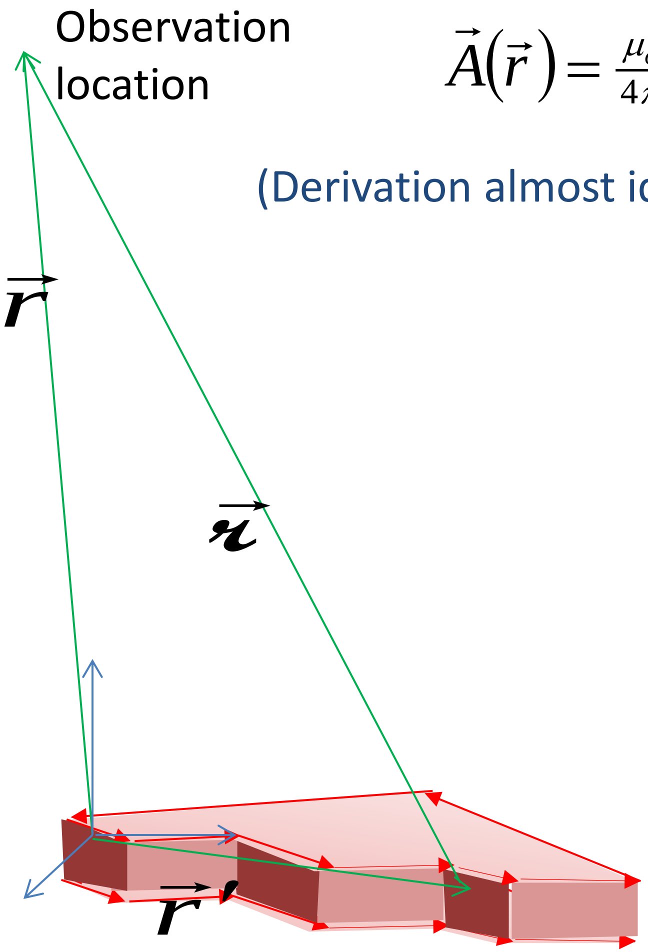
Magnetization

$$\vec{M} \equiv \frac{d\vec{m}}{d\tau'} \qquad \vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int \frac{\vec{M}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau' \qquad \vec{K}_b = \vec{M}\times\hat{n} \qquad \vec{J}_b = \vec{\nabla}\times\vec{M}$$

It may be familiar (without the b-subscripts) that

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int \frac{\vec{J}_b\, d\tau'}{\mathscr{r}} + \frac{\mu_o}{4\pi}\oint \frac{\vec{K}_b\, da'}{\mathscr{r}}$$

(Derivation almost identical to that for polarization's scalar potential)



$\vec{K}_b = \vec{M} \times \hat{n}$

Magnetization $\vec{M} \equiv \frac{d\vec{m}}{d\tau'}$

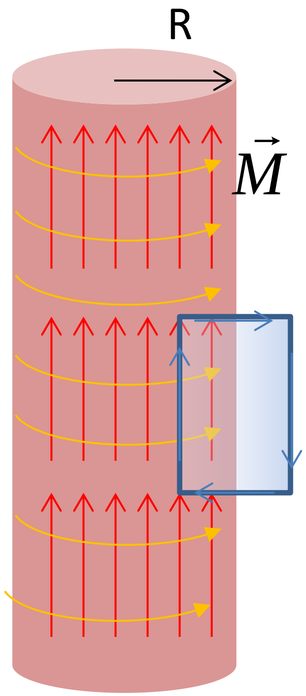
$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{J}_b = \vec{\nabla} \times \vec{M}$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int \frac{\vec{J}_b d\tau'}{\mathscr{r}} + \frac{\mu_o}{4\pi} \oint \frac{\vec{K}_b da'}{\mathscr{r}}$$

$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$

Example: An infinitely long circular cylinder carries a uniform magnetization parallel to its axis. Find the magnetic field inside and outside the cylinder.



$\vec{M} \equiv M\hat{z}$

Before the math, think about the arrangement of microscopic current loops that would give this M.

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = 0 \quad \text{and} \quad \vec{K}_b = \vec{M} \times \hat{n} = (M\hat{z}) \times \hat{s} = M\hat{\phi}$$

Like a solenoid (can you say “bar magnet”)

Amperian loop and argument for (comparatively) uniform and 0 outside

$$\oint \vec{B} \cdot d\vec{l} = \mu_o I_{encl}$$

$$BL = \mu_o ML$$

$$B = \mu_o M$$

$$\vec{B} = \begin{cases} \mu_0 M\hat{z} = \mu_0 \vec{M} & s < R, \\ 0 & s > R. \end{cases}$$

$$\vec{K}_b = \vec{M} \times \hat{n}$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

Magnetization

$$\vec{M} \equiv \frac{d\vec{m}}{d\tau'}$$

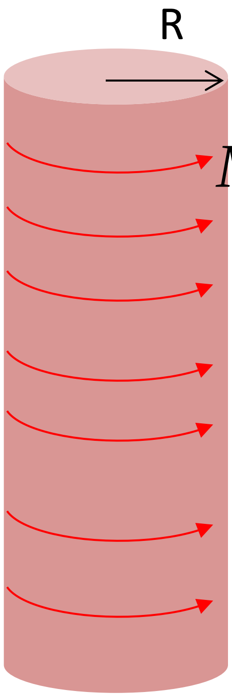
$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int \frac{\vec{J}_b d\tau'}{\mathscr{r}} + \frac{\mu_o}{4\pi} \oint \frac{\vec{K}_b da'}{\mathscr{r}}$$

Exercise: An infinitely long circular cylinder carries a uniform, circumferential magnetization. Find the magnetic field inside and outside the cylinder.

Before the math, think about the arrangement of microscopic current loops that would give this M.



$$\vec{M} \equiv M\hat{\phi}$$

Warning: the math of J is a little subtle.

$\vec{K}_b = \vec{M} \times \hat{n}$

Magnetization $\vec{M} \equiv \frac{d\vec{m}}{d\tau'}$

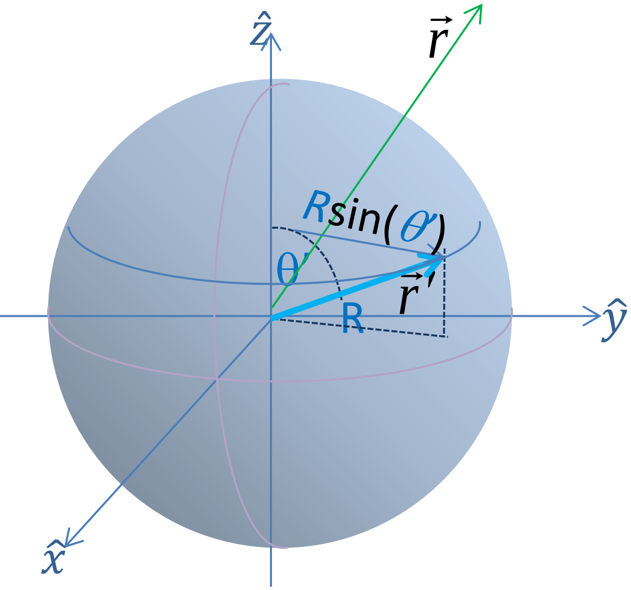
$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{J}_b = \vec{\nabla} \times \vec{M}$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int \frac{\vec{J}_b d\tau'}{\mathscr{r}} + \frac{\mu_o}{4\pi} \oint \frac{\vec{K}_b da'}{\mathscr{r}}$$

$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$

Ex. 6.1: What's the magnetic potential of a sphere with constant magnetization in the z direction?



$$\vec{K}_b = M\hat{z} \times \hat{r} = M \sin\theta' \hat{\phi}$$

Griffiths points out that this has the same form as example 5.11: uniform surface charge density, σ, rotating with angular speed ω.

$$\vec{K} = \sigma\vec{v} = \sigma\left(\omega R \sin\theta' \hat{\phi}\right) = (\sigma\omega R)\sin\theta'\hat{\phi}$$

So you can jump to the conclusion and substitute M in place of $\sigma\omega$R

$$\vec{A}(\vec{r}) = \begin{cases} \frac{\mu_o}{3} \sigma R \vec{\omega} \times \vec{r} & r < R \\ \frac{\mu_o}{3r^3} \sigma R^4 \vec{\omega} \times \vec{r} & r > R \end{cases}$$

becomes

$$\begin{cases} \frac{\mu_o}{3} \vec{M} \times \vec{r} \\ \frac{\mu_o}{3r^3} R^3 \vec{M} \times \vec{r} \end{cases}$$

$\vec{K}_b = \vec{M} \times \hat{n}$

Magnetization

$\vec{M} \equiv \dfrac{d\vec{m}}{d\tau'}$

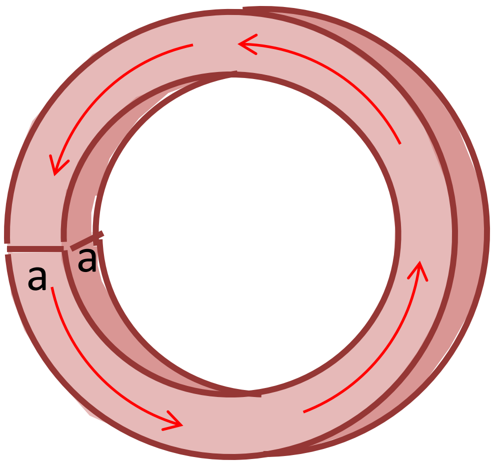
$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{J}_b = \vec{\nabla} \times \vec{M}$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int \frac{\vec{J}_b d\tau'}{\mathscr{r}} + \frac{\mu_o}{4\pi} \oint \frac{\vec{K}_b da'}{\mathscr{r}}$$

$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$

Exercise: An iron rod of length *L* and square cross section (side *a*) is given a uniform longitudinal magnetization M, then is bent around into a circle. Find the magnetic field everywhere.



$$\vec{M} \equiv M\hat{\phi}$$

$\vec{K}_b = \vec{M} \times \hat{n}$

$\vec{J}_b = \vec{\nabla} \times \vec{M}$

Magnetization

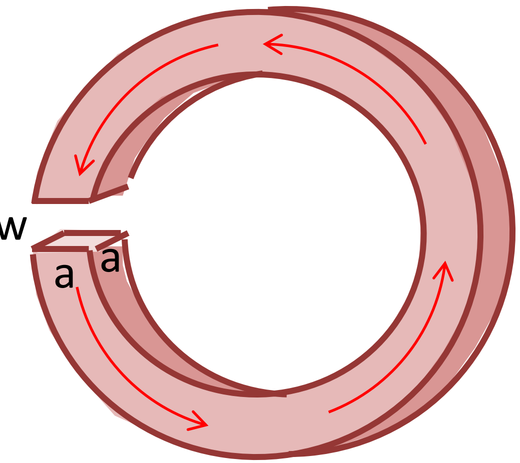
$\vec{M} \equiv \frac{d\vec{m}}{d\tau'}$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int \frac{\vec{J}_b d\tau'}{\mathscr{r}} + \frac{\mu_o}{4\pi} \oint \frac{\vec{K}_b da'}{\mathscr{r}}$$

Example: An iron rod of length *L* and square cross section (side *a*) is given a uniform longitudinal magnetization M, then is bent around into a circle. with a narrow gap (width *w*). Find the magnetic field at the center of the gap, assuming w<<a<<L.



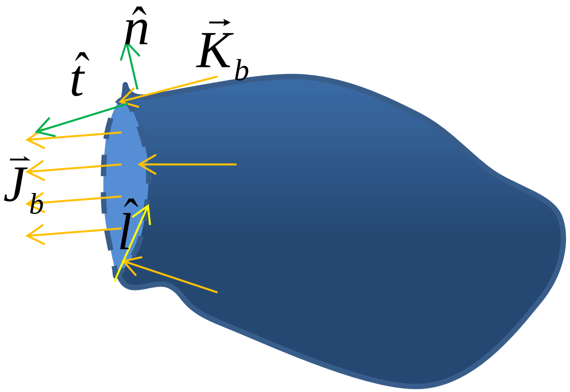
$\vec{M} \equiv M\hat{\phi}$

Magnetization

$$\vec{M} \equiv \frac{d\vec{m}}{d\tau'} \qquad \vec{K}_b = \vec{M} \times \hat{n} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \qquad \vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int \frac{\vec{J}_b d\tau'}{\mathscr{r}} + \frac{\mu_o}{4\pi}\oint \frac{\vec{K}_b da'}{\mathscr{r}} \qquad \vec{\nabla} \times \vec{B} = \mu_o \vec{J}$$

Example: Like bound *charge,* total bound current must be 0 for any shaped object.



$$\vec{I}_b = \int \vec{J}_b \cdot d\vec{a} + \oint \left(\vec{K}_b \cdot \hat{t}\right) dl$$

$$\vec{I}_b = \int \left(\vec{\nabla} \times \vec{M}\right) \cdot d\vec{a} + \oint \left(\left(\vec{M} \times \hat{n}\right) \cdot \hat{t}\right) dl$$

Stokes

$$\vec{I}_b = \oint \vec{M} \cdot d\vec{l} + \oint \left(\left(\vec{M} \times \hat{n}\right) \cdot \hat{t}\right) dl$$

Product Rule 1 $\left(\vec{a} \times \vec{b}\right) \cdot c = \vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$

$$\vec{I}_b = \oint \vec{M} \cdot d\vec{l} + \oint \vec{M} \cdot \left(\hat{n} \times \hat{t}\right) dl$$

$$\left(\hat{n} \times \hat{t}\right) = -\hat{l}$$

$$\vec{I}_b = \oint \vec{M} \cdot d\vec{l} - \oint \vec{M} \cdot d\vec{l}$$

$$\vec{I}_b = 0$$

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