Prep for Exam 2: topics covered, equations encountered

Ch 5. Oscillations

5.1 Hook's Law

$$F_x(x) = -k(x - x_{eq}) \qquad U(x) = \tfrac{1}{2}k(x - x_{eq})^2$$

- Taylor Series
  - For a given potential or force, find the 2nd-order term in the Taylor Series and thus the 'spring stiffness.'
  - $f(x) = f(x_o) + \sum_{n=1} \frac{1}{n!} \left.\frac{d^n f(x)}{dx^n}\right|_{x_o} (x - x_o)^n$

5.2 Simple Harmonic Motion

$$m\ddot{x}(t) = -kx(t)$$

$$\omega \equiv \sqrt{\frac{k}{m}}$$

Use Euler's relations to move between these different representations

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

- The Exponential Solutions $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$
- The Sine and Cosine Solutions $x(t) = B_1\cos(\omega t) + B_2\sin(\omega t)$
- **Phase-Shifted Cosine Solutions** $x(t) = A\cos(\omega t - \delta)$
- Solutions as the Real Part of a Complex Exponential $x(t) = \mathrm{Re}\, Ae^{i(\omega t - \delta)}$
- Energy Considerations

5.3 Damped Oscillations

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0 \qquad x_h(t) = \begin{cases} e^{-\beta t} A\cos(\omega_1 t - \delta) & \text{under damped} \\ e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2 - \omega_o^2}\cdot t} + C_2 e^{-\sqrt{\beta^2 - \omega_o^2}\cdot t}\right) & \text{over damped} \\ e^{-\beta t}(A + Bt) & \text{critically damped} \end{cases} \quad \text{where } \omega_1 = \sqrt{\beta^2 - \omega_o^2}$$

5.4 2-D Oscillators

5.5 Driven Damped Oscillations

$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f(t)$ (in homogeneous)

- Linear Differential Operators: a linear combination of solutions is also a solutions,
- Particular and Homogeneous Solutions: $x = x_h + x_p$

Prep for Exam 2: topics covered, equations encountered

- Complex Solutions for a Sinusoidal Driving Force
- $\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f(t)$ with $f(t) = f_o \sin(\omega_D t)$ $x(t) = A\sin(\omega_D t - \delta) + x_h(t)$
- $A = \dfrac{f_o}{\sqrt{\left(\omega_o^2 - \omega_D{}^2\right)^2 + \left(2\beta\omega_D\right)^2}}$ $\delta = \arctan\left(\dfrac{2\beta\omega_D}{\omega_o^2 - \omega_D{}^2}\right)$
- Resonance
  - $\omega_{res} = \sqrt{\omega_o^2 - 2\beta^2}$, $A^2{}_{\max} = \dfrac{f_o^2}{4\beta^2\left(\omega_o^2 - \beta^2\right)}$
- Width of the Resonance: the Q Factor $\omega_{\frac{1}{2}} \approx \omega_o \mp \beta$
- The Phase of resonance $\delta = \tan^{-1}\left(\dfrac{2\beta\omega_D}{\omega_o^2 - \omega_D{}^2}\right)$

5.7 Fourier Series

$$F(\omega t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

5.8 Fourier Series Solution for the Driven Oscillator

A force like $F(\omega t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$

Gives rise to a solution like $x(t) = x_h(t) + \sum_{n=0}^{\infty} \left(A_{cn} \cos(n\omega_D t - \delta_n) + A_{sn} \sin(n\omega_D t - \delta_n)\right)$

5.9 RMS Displacement Parseval's Theorem

Chapter 6 Calculus of Variations

6.1 Shortest Path and Fermat's Principle

$$S = \int_{s_1}^{s_2} ds = \int_{x_1}^{x_2} \sqrt{1 + \left(y'(x)\right)^2}\, dx$$

$$t = \int_{s_1}^{s_2} \frac{ds}{v(x,y)} = \int_{x_1}^{x_2} \left(\frac{\sqrt{1 + \left(y'(x)\right)^2}}{v(x,y)}\right) dx$$

6.2 Euler-Lagrange Equation

Prep for Exam 2: topics covered, equations encountered

Generally, $F = \int_{s_1}^{s_2} f(x, y, y')dx$ is maximized, minimized, or stationary if

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$S = \int f(r,\phi(r),\phi'(r))dr\ ,\ \ \frac{\partial f}{\partial \phi} - \frac{d}{dr}\frac{\partial f}{\partial \phi'} = 0$$

6.3 Applications of the Euler-Lagrange Equation

Shortest path on sphere, on cylinder, minimum potential energy curve, minimum time path in gravitational field.

Maximum and Minimum vs. Stationary

6.4 More than Two Variables

$$S = \int_{u_1}^{u_2} f\left[x(u), y(u), z(u), x'(u), y'(u), z'(u), u\right]du$$

To max/min-imize (or find stationary),

$$\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0, \qquad \frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0, \qquad \text{and} \qquad \frac{\partial f}{\partial z} - \frac{d}{du}\frac{\partial f}{\partial z'} = 0.$$

Needn't be Cartesian, for example,

Chapter 7 Lagrange's Equations

7.1 Lagrange's Equations for Unconstrained Motion

- $\mathcal{L} = T - U$
- Regardless of what coordinates we express it in terms of
- $\mathcal{L} \equiv T(t, q_i(t), \dot{q}_1(t),\ldots q_N(t), \dot{q}_N(t)) - U(t, q_i(t), \dot{q}_1(t),\ldots q_N(t), \dot{q}_N(t))$
-
- $\int \mathcal{L}(t, q_i(t), \dot{q}_1(t),..q_N(t), \dot{q}_N(t))dt$
- Of course, that's equivalent to saying that the Lagrangian satisfies the differential equations

Prep for Exam 2: topics covered, equations encountered

- $\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$ for all the individual coordinates.
- Several Unconstrained Particles

7.2 Constrained Systems; an example

At least one 'degree of freedom' can be rephrased in terms of another.

Spring-mass-pulley-hanging mass

Mass on parabolic wire

Mass on sphere

Pendulum hanging form cart

Bob hanging from orbiting disc

Double pendulum

Block sliding down slipping slope

Mass on spinning parabolic wire

7.3 Constrained Systems in General

- Degrees of Freedom

7.4 Proof of Lagrange's Equations with Constraints

- The Action Integral is Stationary at the Right Path
- The Final Proof

7.5 Examples of Lagrange's Equations

7.6 Generalized Momenta and Ignorable Coordinates

7.7 Conclusions

7.8 More about Conservation Laws