Wed., 11/28 Thurs. 11/29 Fri., 11/30	**9.8-.9** Free Fall & Coriolis, Foucault Pendulum **10.1-.2** Center of Mass & Rotation about a Fixed Axis	**HW9c** (9.25, 9.27)
Mon., 12/3 Tues. 12/4 Wed., 12/5 Thurs. 12/6 Fri., 12/7	**10.3-.4** Rotation about any Axis, Inertia Tensor Principle Axes **10.5-.6** Finding Principle Axes, Precession **10.7-.8** Euler's Equations	**HW10a** (10.6-.22) **HW10b** (10.36, 10.39)
Mon., 12/10	**Review for Final**	**Project**

Equipment:

- Globe
- Ball with coordinate axes
- Turntable with paper taped to it
- Pendulum spinning on turn table
- Simulation for HW 24

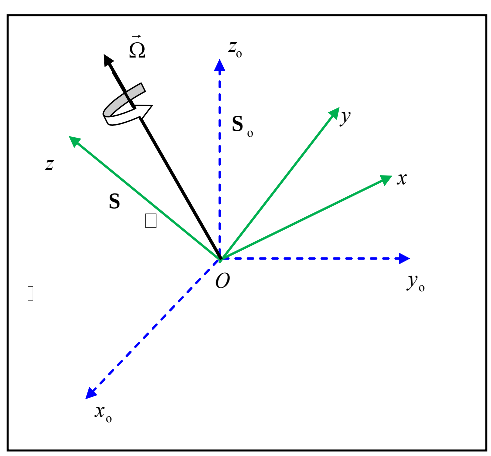
Non-inertial Frames: Rotating

Last time we learned that when one frame is rotating relative to the other, say, the Earth, relative to the 'fixed stars', then velocity and acceleration measurements made in the two frames are related by

$$\dot{\vec r} = \dot{\vec r}_o - \vec V_f(\vec r) \quad \text{where } \vec V_f(\vec r) = \vec\Omega\times\vec r \text{, i.e., } \vec V_f(\vec r) = r_{axis}\Omega\hat\phi$$

And

$$\ddot{\vec r} = \ddot{\vec r}_o - \left(\vec A_f(\vec r) + \vec A_{Corr}(\dot{\vec r})\right) \text{ where } \vec A_f = \vec A_{cent} = \left(\vec r\times\vec\Omega\right)\times\vec\Omega = \left(-r_{axis}\dot\phi^2\right)\hat r_{axis} \text{ and } \vec A_{corr} = -2\dot{\vec r}\times\vec\Omega$$



Of course, Newton's 2nd Law applies only in an inertial frame

$$\ddot{\vec{r}}_o = \frac{\vec{F}_{net}}{m}$$

So,

$$\ddot{\vec{r}} = \frac{\vec{F}_{net}}{m} - \left(\vec{A}_f(\vec{r}) + \vec{A}_{Corr}(\dot{\vec{r}})\right)$$

Fictitious Inertial / Frame Force

$$m\ddot{\vec{r}} = \vec{F}_{net} + \vec{F}_{frame}$$

$$\vec{F}_{frame} \equiv -m\left(\vec{A}_f(\vec{r}) + \vec{A}_{Corr}(\dot{\vec{r}})\right) = \vec{F}_{centripetal} + \vec{F}_{coriolis} \text{ where}$$

$$\vec{F}_{cent} = -m\vec{A}_{cent} = m\left(\vec{\Omega}\times\vec{r}\right)\times\vec{\Omega} = \left(mr_{axis}\dot{\phi}^2\right)\hat{r}_{axis} \text{ and } \vec{F}_{corr} = -m\vec{A}_{corr} = m2\vec{\Omega}\times\dot{\vec{r}}$$

Today and tomorrow, we'll look at some effects of the centrifugal force:

$$\vec{F}_{\mathrm{cf}} = m\left(\vec{\Omega}\times\vec{r}\right)\times\vec{\Omega},$$

and the Coriolis force:

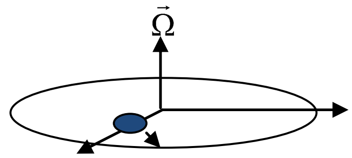
$$\vec{F}_{\mathrm{cor}} = 2m\dot{\vec{r}}\times\vec{\Omega}.$$

How can we observe an effect of the Coriolis force on the motion of an object near the earth? (We already considered hurricanes, but they are complicated systems of particles.) Our calculations will be done for the northern hemisphere.

<u>Both Forces</u>:

To get a little practice with a relatively simple situation, let's consider the motion of a frictionless puck on a horizontal, rotating turntable. Compared to the spinning of the turntable on its own axis, the spinning of the room (sitting on the face of a spinning Earth) is negligible, so we'll treat the room as an inertial frame. Of course, in the inertial frame the puck will simply move in a straight line because there is no net force. A (noninertial) rotating observer may observe more complicated motions which will be explained by the centrifugal and the Coriolis forces.

<u>Example #</u>: Prob. 9.20 (background for 9.24) Suppose a frictionless puck moves on a horizontal turntable rotating counterclockwise (viewed from above) at an angular speed Ω. Write down the equations of motion for the puck in the rotating system if the puck starts at an initial position $\vec{r}_\mathrm{i} = (x_\mathrm{i}, 0)$ with an initial velocity $\vec{v}_\mathrm{i} = (v_{x\mathrm{i}}, v_{y\mathrm{i}})$ as measured in the rotating frame. Ignore Earth's rotation!



In an inertial frame there is no net force on the puck, so

$m\ddot{\vec{r}}_o = \vec{F}_\mathrm{net} = 0$ and *we'd* see the puck moving with constant velocity / in a straight line.

How would it look to a little bug ridding on the turntable? In that noninertial frame that rotates with the turntable, Newton's second law is:

$$m\ddot{\vec{r}} = \vec{F}_\mathrm{cf} + \vec{F}_\mathrm{cor} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} + 2m\dot{\vec{r}} \times \vec{\Omega}.$$

Taking the angular velocity of the turntable to be $\vec{\Omega} = (0,0,\Omega)$, the position is $\vec{r} = (x,y,0)$. Calculate the cross products:

$$\text{I do: } \vec{\Omega} \times \vec{r} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \Omega \\ x & y & 0 \end{vmatrix} = (-\Omega y, \Omega x, 0),$$

$$\text{They do: } (\vec{\Omega} \times \vec{r}) \times \vec{\Omega} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\Omega y & \Omega x & 0 \\ 0 & 0 & \Omega \end{vmatrix} = (\Omega^2 x, \Omega^2 y, 0),$$

$$\textbf{They do: } \dot{\vec{r}} \times \vec{\Omega} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & 0 \\ 0 & 0 & \Omega \end{vmatrix} = (\Omega\dot{y}, -\Omega\dot{x}, 0).$$

So the equation of motion gives (dividing out the mass):

$$(\ddot{x}, \ddot{y}, 0) = (\Omega^2 x, \Omega^2 y, 0) + (2\Omega\dot{y}, -2\Omega\dot{x}, 0),$$

or the equations for the x and y components are:

$$\ddot{x} = \Omega^2 x + 2\Omega\dot{y} \quad \text{and} \quad \ddot{y} = \Omega^2 y - 2\Omega\dot{x}.$$

A trick for solving <u>both</u> of these coupled differential equations at the same time (see Sect. 2.7) is to define $\eta = x + iy$.

Complex notation: We did something like this back when we were dealing with damped, driven harmonic oscillators. In this case, the interpretation is even simpler: this is essentially handy notation for the 2-D vector ρ, where the *i* plays the role of y-hat. At the end of the day, we'll be able to break our solution back apart into the x and y components by using this fact.

If we add *i* times the $\ddot{y}$-equation to the $\ddot{x}$-equation, we get:

$$\ddot{x} + i\ddot{y} = \Omega^2 (x + iy) + 2\Omega(-i\dot{x} + \dot{y}) = \Omega^2 (x + iy) - 2i\Omega(\dot{x} + i\dot{y}),$$

$$\ddot{\eta} = \Omega^2\eta - 2i\Omega\dot{\eta}.$$

This looks an awful lot like the damped harmonic oscillator (aside from that factor of *i* and the lack of a negative sign on the linear term). So we can guess the basic form of the solution that we'd guessed in that case.

Since this is a linear, differential equation, guess the solution $\eta = e^{-i\alpha t}$, which gives the auxiliary equation:

They do: $$-\alpha^2 = \Omega^2 - 2\Omega\alpha,$$

$$\Omega^2 - 2\Omega\alpha + \alpha^2 = (\Omega - \alpha)^2 = 0.$$

This implies that $\alpha = \Omega$. There is only one solution for α, so we need a second solution (the differential equation is second order). This sounds a lot like the problem with the critically damped simple harmonic oscillator. So we've got a good chance that a similar solution will work. Just as with critical damping, you can check that in addition to $e^{-i\Omega t}$, $te^{-i\Omega t}$ is a solution, so the general solution is:

$$\eta(t) = e^{-i\Omega t}(C_1 + C_2 t),$$

where the coefficients may be <u>complex</u>.

Impose Initial Conditions

$\vec{r}_\mathrm{i} = (x_\mathrm{i}, 0, 0)$ and $\dot{\vec{r}}_\mathrm{i} = (v_{x\mathrm{i}}, v_{y\mathrm{i}}, 0)$ or $\eta_\mathrm{i} = x_\mathrm{i}$ and $\dot{\eta}_\mathrm{i} = v_{x\mathrm{i}} + iv_{y\mathrm{i}}$.

The first condition implies that

$$C_1 = x_\mathrm{i}$$

and the derivative is:

$$\dot{\eta}(t) = -i\Omega e^{-i\Omega t}(x_i + C_2 t) + C_2 e^{-i\Omega t},$$

So the second condition implies:

$$\dot{\eta}(0) = C_2 - i\Omega x_i = v_{x\mathrm{o}} + iv_{y\mathrm{o}},$$

$$C_2 = v_{x\mathrm{i}} + i(v_{y\mathrm{i}} + \Omega x_i).$$

This gives:

$$\eta(t) = e^{-i\Omega t}\left[x_\mathrm{i} + v_{x\mathrm{i}} t + i(v_{y\mathrm{i}} + \Omega x_\mathrm{i})t\right] = (\cos\Omega t - i\sin\Omega t)\left[x_\mathrm{i} + v_{x\mathrm{i}} t + i(v_{y\mathrm{i}} + \Omega x_\mathrm{i})t\right].$$

The real part of η is $x(t)$ and the imaginary part is $y(t)$, which gives (Eq. 9.72):

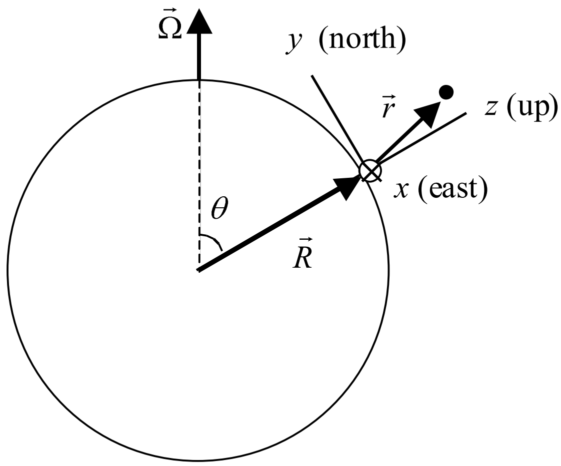
They do: $x(t) = (x_i + v_{xi}t)\cos\Omega t + (v_{yi} + \Omega x_i)t\sin\Omega t$,

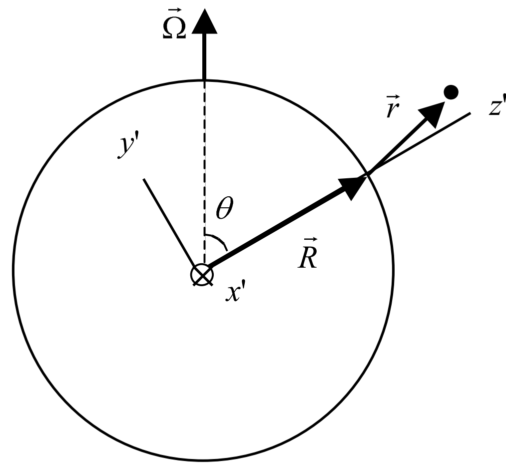
$y(t) = -(x_i + v_{xi}t)\sin\Omega t + (v_{yi} + \Omega x_i)t\sin\Omega t$.

You will explore (computationally) the behavior of the motion for different initial velocities in the homework (Prob. 9.24).

<u>Free Fall</u>:

We will use the coordinate axes *x*, *y*, and *z* with the origin on the surface of the earth at the colatitude θ (below on the left). Those coordinates point in the same directions as rotating coordinate axes x', y', and z' with the origin at the center of the earth (below on the right).





The position of the particle can be written as $\vec{R} + \vec{r}$, where $\vec{R}$ is a vector from the center of the earth to a point on the surface at colatitude θ and $\vec{r}$ is the position relative the point on the surface. We'll assume that the experiment takes place near the surface of the earth, so $r << R$ and $\vec{R} + \vec{r} \approx \vec{R}$. The centrifugal force is approximately:

$$\vec{F}_{\text{cf}} \approx m\left(\vec{\Omega} \times \vec{R}\right) \times \vec{\Omega}.$$

A plumb line will point along the observed $\vec{g}$, which is $\vec{g} \approx \vec{g}_{\text{o}} + \left(\vec{\Omega} \times \vec{R}\right) \times \vec{\Omega}$ (as discussed last time). The direction of $\vec{g}$ defines the direction of the *z* axis. We will use the colatitude θ as the angle between the *z* axis and the angular momentum vector $\vec{\Omega}$, ignoring the slight correction discussed last time.

Newton's second law in the rotating frame gives:

$$m\ddot{\vec{r}} \approx m\vec{g}_{\text{o}} + 2m\dot{\vec{r}} \times \vec{\Omega} + m\left(\vec{\Omega} \times \vec{R}\right) \times \vec{\Omega},$$

$$\ddot{\vec{r}} = \vec{g} + 2\dot{\vec{r}} \times \vec{\Omega}.$$

The angular velocity in the rotating coordinate system is:

$$\vec{\Omega} = \left(0, \Omega\sin\theta, \Omega\cos\theta\right),$$

so the cross product is:

$$\dot{\vec{r}} \times \vec{\Omega} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & \Omega\sin\theta & \Omega\cos\theta \end{vmatrix} = \left(\dot{y}\Omega\cos\theta - \dot{z}\Omega\sin\theta, -\dot{x}\Omega\cos\theta, \dot{x}\Omega\sin\theta\right).$$

The components of the equation of motion are:

$$\begin{aligned} \ddot{x} &= 2\Omega\left(\dot{y}\cos\theta - \dot{z}\sin\theta\right), \\ \ddot{y} &= -2\Omega\dot{x}\cos\theta, \\ \ddot{z} &= -g + 2\Omega\dot{x}\sin\theta. \end{aligned}$$

Suppose that an object is dropped from rest at $x = y = 0$ and $z = h$.

The book goes about making iterative approximations, in the same vein as when we were looking for the range of a projectile subject to drag. Alternatively, this problem (and the analogous one for a charge moving in both an electric and a magnetic field) can be solved exactly.

Here are notes on both approaches: the iterative approximation and then the exact.

Iterative Approximation solution

As a "zeroeth order" approximation, we can drop all terms containing Ω. This gives:

$$\ddot{x} = \ddot{y} = 0 \quad \text{and} \quad \ddot{z} = -g,$$

so integrating twice gives:

$$\dot{x} = \dot{y} = x = y = 0, \quad \dot{z} = -gt, \quad \text{and} \quad z = h - \tfrac{1}{2}gt^2 .$$

The object will land ($z = 0$) at about the time:

$$t = \sqrt{2h/g}.$$

To get a "**first order**" approximation, put the zeroeth order approximation for the *z* component of the velocity in the original equations of motion to get:

$$\begin{aligned} \ddot{x} &= +2\Omega gt\sin\theta, \\ \ddot{y} &= 0, \\ \ddot{z} &= -g. \end{aligned}$$

Qualitativley: remember that x points East, so from the inertial perspective, a ball that's "dropped" looks like it's thrown with an Eastward initial velocity. As the ball fall's, that Eastward initial is comparatively larger and larger than the eastward velocity of buildings, trees, etc. at its smaller and smaller radius – so it looks like it's accelerating East.

The integrating the *x* equation twice gives:

$$\dot{x} = \Omega gt^2\sin\theta,$$

$$x = \tfrac{1}{3}\Omega gt^3\sin\theta.$$

When the object lands:

$$x = \tfrac{1}{3}\Omega g\left(\frac{2h}{g}\right)^{3/2}\sin\theta = \tfrac{2}{3}\Omega\sqrt{\frac{2h^3}{g}}\sin\theta.$$

At a colatitude (& latitude) of $\theta = 45^\circ$ and height of 100 meters, the eastward deflection would be:

$$x = \tfrac{2}{3}\left(7.3\times10^{-5}\ \text{rad/s}\right)\sqrt{\frac{2(100\ \text{m})^3}{9.8\ \text{m/s}^2}}\sin 45^\circ = 0.0155\ \text{m} = 1.55\ \text{cm}.$$

To get a "second order" approximation for the y component of the acceleration (it is zero in the first order), substitute the first order approximation for the x component of the velocity into the original equation for $\ddot{y}$:

$$\ddot{y} = -2\Omega^2 g t^2 \sin\theta\cos\theta\,.$$

To keep terms of order Ω^2 for y, we'll ignore the small correction to the z component. Integrate the equation twice to get:

$$\dot{y} = -\tfrac{2}{3}\Omega^2 g t^3 \sin\theta\cos\theta,$$

$$y = -\tfrac{1}{6}\Omega^2 g t^4 \sin\theta\cos\theta.$$

When the object lands:

$$y = -\tfrac{1}{6}\Omega^2 g\left(\frac{2h}{g}\right)^2\sin\theta\cos\theta = -\frac{2\Omega^2 h^2}{3g}\sin\theta\cos\theta.$$

At a colatitude (& latitude) of $\theta = 45^\circ$ and height of 100 meters, the southward deflection would be:

$$|y| = \frac{2\left(7.3\times10^{-5}\ \text{rad/s}\right)^2(100\ \text{m})^2}{3\left(9.8\ \text{m/s}^2\right)}\sin 45^\circ\cos 45^\circ = 1.8\times10^{-6}\ \text{m} = 1.8\times10^{-4}\ \text{cm}.$$

Gradients in the gravitational field of the earth can also contribute to the southward deflection of a falling object, but this calculation gives the right order of magnitude.

Exact Solution

Returning to the system of equations that we want to solve:

$$\begin{aligned}
&\ddot{x} = 2\Omega\left(\dot{y}\cos\theta - \dot{z}\sin\theta\right), & & \dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0,\\
&\ddot{y} = -2\Omega\dot{x}\cos\theta, & \text{with the initial conditions that}\quad & x(0) = y(0) = 0,\\
&\ddot{z} = -g + 2\Omega\dot{x}\sin\theta. & & z(0) = h.
\end{aligned}$$

If we integrated the second and third equation we have

$$\dot{y} = -2\Omega x\cos\theta,$$

$$\dot{z} = -gt + 2\Omega x\sin\theta.$$

Then substituting these expressions back into the first equation, we have

$$\ddot{x} = 2\Omega\left(-2\Omega x\cos\theta\cos\theta - \left(-gt + 2\Omega x\sin\theta\right)\sin\theta\right) = \left(g2\Omega\sin\theta\right)t - 4\Omega^2\left(\cos^2\theta + \sin^2\theta\right)x$$
$$\ddot{x} + \left(4\Omega^2\right)x - \left(g2\Omega\sin\theta\right)t = 0$$

So our task is just to solve this differential equation; once we have an expression for x, we can easily take its derivative and thus find the expression for the accelerations in y and z as well.

A good guess would have the form

$$x(t) = A\sin(\omega t - \delta) + Bt$$

Plugging that in, we find that it works for all values of t if

$$B = \frac{g\sin\theta}{2\Omega} \text{ and } \omega = 2\Omega$$

Imposing the initial condition that x(0)=0 tells us that $\delta = 0$.

Imposing the initial condition that $\dot{x}(0) = 0$ then tells us

$$\dot{x}(0) = 0 = \omega A + B \Rightarrow A = -\frac{B}{\omega} = -\frac{g\sin\theta}{4\Omega^2}$$

Putting this all together, we have

$$x(t) = \frac{g\sin\theta}{4\Omega^2}\left(2\Omega t - \sin\left(2\Omega t\right)\right)$$

Before proceeding, note that if the argument of sine is quite small, you can replace it with the first few terms in its Taylor series, this gives

$$x \approx \frac{\sin\theta}{2\Omega} g\left(t - \frac{1}{2\Omega}\left(2\Omega t - \tfrac{1}{6}\left(2\Omega t\right)^3\right)\right) = \frac{\sin\theta}{2\Omega} g\left(t - t + \tfrac{2}{3}\Omega^2 t^3\right) = \tfrac{1}{3}\Omega g t^3 \sin\theta$$

This is a tad deceptive since our "g" itself is dependent on the co-latitude. In particular, $g = \sqrt{g_{rad}^2 + g_{\tan}^2} = \sqrt{g_o^2 + \left(R\Omega^2 - 2g_o\right)R\Omega^2\sin^2\theta}$ based on equations 9.45 and 9.47

With this in hand, we then know that

$$\dot{y} = -\frac{g\sin\theta\cos\theta}{2\Omega}\left(2\Omega t - \sin\left(2\Omega t\right)\right)$$

$$\dot{z} = -gt + \frac{g\sin^2\theta}{2\Omega}\left(2\Omega t - \sin\left(2\Omega t\right)\right) = -g\left(\cos^2\theta\right)t - \frac{g\sin^2\theta}{2\Omega}\left(\sin\left(2\Omega t\right)\right)$$

So,

$$y = -\frac{g\sin\theta\cos\theta}{2\Omega}\left(\Omega t^2 + \frac{1}{2\Omega}\cos\left(2\Omega t\right)\right),$$

$$z = h - \tfrac{1}{2}g\left(\cos^2\theta\right)t^2 - \frac{g\sin^2\theta}{4\Omega^2}\left(1 - \cos\left(2\Omega t\right)\right)$$

Eastward deflections were measured by several experiments between about 1800 and 1900. The following is a summary of experiments from M.S. Tiersten and H. Soodak, *Am. J. Phys.* **68** (2), 129-142 (2000). In their notation, y is eastward and x is southward. The southward deflection was too small to measure in the experiments.

Table I. Summary of recorded deflections.

Observer	Date	θ (deg)[a]	h (m)	Δy (cm)[b]	Δy_{Cor} (cm)[c]	Δx (cm)[b]	Δx_{Cor} (cm)[c]	$g^*_{x,z}$ (E)[d]	# of drops
Guglielmini	1791	45.5	78.3	1.9	1.08	1.2	1.1×10^{-4}	2.3×10^{4}	16
Benzenberg	1802	36.5	76.3	0.9	0.87	0.34	1.0×10^{-4}	6.8×10^{3}	32
Reich[e]	1831	42	158.5	2.8	2.94	0.44	4.5×10^{-4}	2.0×10^{3}	106
Rundell[e]	1848	39.5	400	?	11.2	$\approx$25	2.8×10^{-4}	1.8×10^{5}	$>\approx$50
Hall	1902	48	23	0.15	0.18	0.005	0.95×10^{-5}	1.0×10^{3}	948
Flammarion	1903	41	68	0.63	0.81	-0.16	0.83×10^{-4}	-4.1×10^{3}	144

[a]The angle given is the value of the geocentric polar angle, which is very close to θ.
[b]These columns list the measured values.
[c]These columns are evaluated using Eqs. (26b) and (26e) with $g_0=9.8\ \text{m/s}^2$.
[d]$g^*_{x,z}$ is the value of the effective field gradient required in Eq. (26a)–(26c) to obtain the measured value of Δx. It is given is units of Eötvös, $E=10^{-9}\ \text{s}^{-2}$.
[e]The experiments of Reich and Rundell involved drops down mine shafts.

An alternative approach to the calculation above is to think of the path of the object as an orbit in an inertial reference frame. The rotation of the Earth has to be taken into account after determining the orbit to find the path seen by a rotating observer. (The orbital motion of the Earth about the Sun will be insignificant during this type of experiment.)

<u>The Foucault Pendulum</u>: What's special about it?

> "Foucault's wonderful discovery was the realization that the small effects of the Coriolis force could be greatly multiplied by using a pendulum. What a wonderful day it must have been for Foucault when he noticed that the rightward deflection of one swing would *not* be undone on the return swing; the effects would *accumulate*!" – R.H. Romer, *Am. J. Phys.* **51** (8), 683 (1983).
>
> "Thus the pendulum has the advantage that the effects [of the Coriolis force] accumulate, and thus the effect moves from the domain of theory to that of observation." – Léon Foucault

A long, spherical pendulum (<u>not</u> constrained to move in a plane) that has a very small amplitude of oscillation moves approximately in a horizontal plane. We'll use the same coordinates that we did for free fall. We can ignore the displacement ($z \approx 0$) and velocity ($\dot{z} \approx 0$) in the vertical direction.

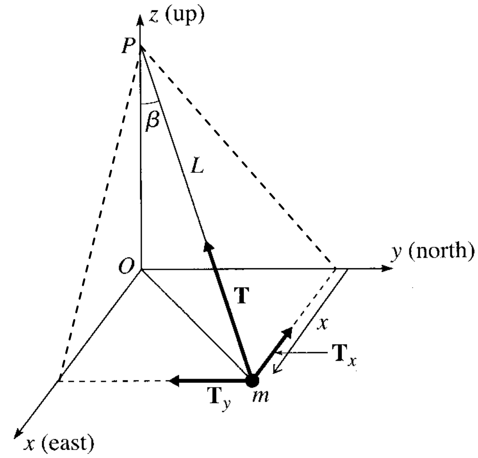
Before we get started solving this, let's think about what we'd expect, looking at the situation from the inertial / non-rotating frame. To make it really simple, we'll consider a pendulum at the North pole. The pendulum swings back and forth rather oblivious to the fact that the Earth beneath it is rotating. Focusing on just the motion of the bob in the plane, we'd see

$$\vec{\rho}_o(t) = \left(A\cos\omega_o t\right)\hat{x}_o$$

But the rotating frame's axes are rotating relative to the inertial ones; relative to *those* axes, we'd say

$$\vec{\rho}(t) = A\cos\omega_0 t\,\left(\cos\Omega_z t\,\hat{x} - \sin\Omega_z t\,\hat{y}\right)$$

Now we're going to prove this guess right.



For small oscillations, $T \approx T_z \approx mg$ because the acceleration of the bob is very small and $L >> x, y$. The *x* and *y* components of the tension are proportional to the displacements:

$$\frac{-T_x}{T} = \frac{x}{L} \quad \text{and} \quad \frac{-T_y}{T} = \frac{y}{L}$$

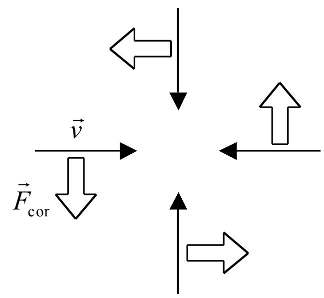
so

$$T_x = -\left(\frac{x}{L}\right)T \approx -mgx/L \quad \text{and} \quad T_y = -\left(\frac{y}{L}\right)T \approx -mgy/L\,.$$

The equations of motion (with an *m* factored out) are the same as for free fall, except for the addition of the tension:

$$\ddot{x} = T_x/m + 2\Omega\left(\dot{y}\cos\theta - \dot{z}\sin\theta\right) \approx -gx/L + 2\Omega\dot{y}\cos\theta,$$
$$\ddot{y} = T_y/m - 2\Omega\dot{x}\cos\theta = -gy/L - 2\Omega\dot{x}\cos\theta.$$

The horizontal components of the Coriolis force when a pendulum is swinging in the Northern hemisphere are shown below (use the component equations above and recall the discussion of hurricanes yesterday).



Define the natural frequency of the pendulum $\omega_o = g/L$ and the z component of the earth's angular momentum $\Omega_z = \Omega\cos\theta$ to get:

$$\ddot{x} - 2\Omega_z\dot{y} + \omega_o^2 x = 0,$$
$$\ddot{y} + 2\Omega_z\dot{x} + \omega_o^2 y = 0.$$

The solution can be found by defining the complex function $\eta = x + iy$. Add the first equation and i times the second one to get:

$$(\ddot{x} + i\ddot{y}) + 2\Omega_z(i\dot{x} - \dot{y}) + \omega_o^2(x + iy) = 0,$$

$$\ddot{\eta} + 2i\Omega_z\dot{\eta} + \omega_o^2\eta = 0.$$

Guess that the solution will be of the form $\eta = e^{-i\alpha t}$, which gives the auxiliary equation:

$$-\alpha^2 + 2\Omega_z\alpha + \omega_o^2 = 0.$$

$$\alpha^2 - 2\Omega_z\alpha - \omega_o^2 = 0.$$

The solution to the quadratic equation is:

$$\alpha = \frac{2\Omega_z \pm \sqrt{4\Omega_z^2 - 4(1)(-\omega_o^2)}}{2(1)} = \Omega_z \pm \sqrt{\Omega_z^2 + \omega_o^2} \approx \Omega_z \pm \omega_o,$$

because $\Omega_z << \omega_o$. The general solution is:

$$\eta(t) = C_1 e^{-i(\Omega_z - \omega_o)t} + C_2 e^{-i(\Omega_z + \omega_o t)} = e^{-i\Omega_z t}\left(C_1 e^{i\omega_o t} + C_2 e^{-i\omega_o t}\right),$$

so:

$$\dot{\eta}(t) = -i(\Omega_z - \omega_o)C_1 e^{-i(\Omega_z - \omega_o)t} + -i(\Omega_z + \omega_o)C_2 e^{-i(\Omega_z + \omega_o t)}.$$

If we choose the initial conditions $x_o = A$, $y_o = 0$, and $v_{xo} = v_{yo} = 0$, then the initial conditions for η are:

$$\eta(0) = A \quad \text{and} \quad \dot{\eta}(0) = 0.$$

These are approximately satisfied ($\Omega_z << \omega_o$) if:

$$C_1 + C_2 = A \quad \text{and} \quad C_1 - C_2 = 0.$$

The coefficients are $C_1 = C_2 = A/2$ and the solution is:

$$\eta = e^{-i\Omega_z t}\left(A\cos\omega_0 t\right) = \left(\cos\Omega_z t - i\sin\Omega_z t\right)A\cos\omega_0 t = x(t) + iy(t).$$

So,

$$\eta = e^{-i\Omega_z t}\left(A\cos\omega_0 t\right) = \left(\cos\Omega_z t + i\sin\Omega_z t\right)A\cos\omega_0 t =$$
$$x(t) = A\cos\omega_0 t\cos\Omega_z t$$
$$y(t) = -A\cos\omega_0 t\sin\Omega_z t$$
$$\vec{\rho}(t) = A\cos\omega_0 t\left(\left(\cos\Omega_z t\right)\hat{x} - \left(\sin\Omega_z t\right)\hat{y}\right)$$

This last expression may be the easiest to interpret: the pendulum swings back and forth at its natural frequency while appearing to rotate clockwise relative to the reference frame which is itself rotating counterclockwise.

The amplitude of the oscillation is A, the frequency of the oscillation is ω_0, and $\boxed{\Omega_z = \Omega\cos\theta}$ is the frequency of the rotation of the direction the pendulum's swing. The angle between the direction of swing and the x axis is $\boxed{\Omega_z t}$. The angular speed of the earth's rotation is $\Omega = 360^\circ/\text{day}$, so at the North Pole ($\theta = 0$) the pendulum rotates once a day. At a latitude of 42° (colatitude $\theta = 48^\circ$):

$$\Omega_z = \Omega\cos 48^\circ \approx \tfrac{2}{3}\left(360^\circ/\text{day}\right) = 240^\circ/\text{day} = 10^\circ/\text{hour}.$$

<u>Demo</u>: Pendulum on a rotating platform. The pendulum continues to spin in the same axis in an inertial frame, but changes direction in the rotating frame. The analogy with the Foucault pendulum is <u>not</u> perfect because the rate of change in the direction of the pendulum's swing (in the rotating frame) does <u>not</u> depend on its location on the turntable. Also, the force on the pendulum is always in the same direction in this case, but it changes as a Foucault pendulum goes around the earth.