Wed., 11/17 Thurs. 11/18 Fri., 11/19	**8.6-8** Unbounded and Changing Orbits **9.1-.3** Noninertial Frames: Acceleration, Tides, Angular Velocity	Ch 6 30-min test Ch 7 30-min test
Mon., 11/22 Tues.,11/23	**9.4-.5** Noninertial Frames: Time derivatives, Newton's 2^{nd}.	Ch 5 30-min test **HW8**

Review/Summary:

Key Equations

$$l = \left|\vec{r} \times \mu\dot{\vec{r}}\right| \qquad U_{eff} = U + \frac{l}{2\mu r^2} \qquad r = \frac{1}{u} \qquad u''(\phi) = -u(\phi) - \frac{\mu}{l^2 u(\phi)^2} F$$

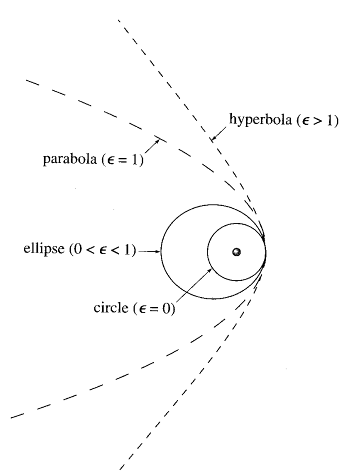
If $F = -\gamma / r^2$

$$r(\phi) = \frac{c}{(1 + \varepsilon\cos(\phi + \delta))} \qquad E = \frac{\gamma^2\mu}{2l^2}(\varepsilon^2 - 1) \qquad c = \frac{\ell^2}{\gamma\mu}\gamma$$

All of the "Kepler orbits" are described by $r(\phi) = \dfrac{c}{(1 + \varepsilon\cos\phi)}$.

Eccentricity	Energy	Shape of Orbit
$\varepsilon = 0$	$E < 0$	circle
$0 < \varepsilon < 1$	$E < 0$	ellipse
$\varepsilon = 1$	$E = 0$	parabola
$\varepsilon > 1$	$E > 0$	hyperbola

These are all "conic sections" – they are shapes that you get when you slice a cone different ways. The four different types of orbits are shown below ($\phi = 0$ is horizontal to the right).



In the equation $r(\phi)=\dfrac{c}{(1+\varepsilon\cos\phi)}$:

$$c=\frac{\ell^2}{\gamma\mu},\qquad \gamma=Gm_1m_2,\qquad \text{and}\qquad \mu=\frac{m_1m_2}{m_1+m_2}.$$

How can you find ℓ and ε?

1. Total angular momentum ($\ell=\left|\vec{r}\times\mu\dot{\vec{r}}\right|$) is easiest to find when $\vec{r}\perp\dot{\vec{r}}$.

   - Where does that occur?

     Point of closest approach (perigee) or farthest point (apogee) – the latter only exists for bounded orbits.

   - What is the value of ℓ at these points?

     Only at these points: $\ell=\mu rv$, where $v=\left|\dot{\vec{r}}\right|$ is the speed of one object relative to the other. At other points on the orbit, it would be much more difficult to determine the angle between $\vec{r}$ and $\dot{\vec{r}}$.

2. How can you find the eccentricity ε (after finding c)?

   Need to know the separation r at any angle ϕ. It is particularly easy to use the perigee or apogee beacuase:

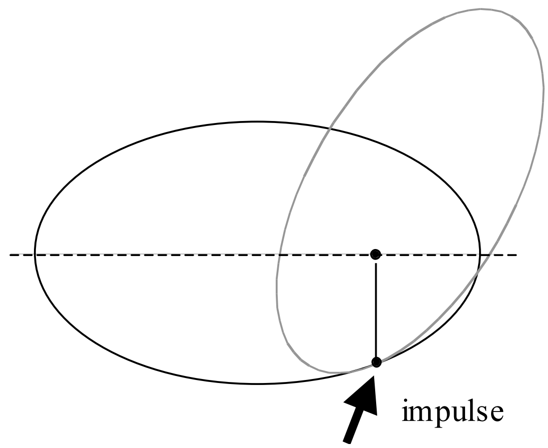
   $$r_{\min}=c/(1+\varepsilon)\qquad\qquad \text{and}\qquad r_{\max}=c/(1-\varepsilon)$$

Changes of Orbit:

The most general form of the equation of orbit is:

$$r_1(\phi) = \frac{c_1}{1+\varepsilon_1\cos(\phi-\delta_1)}.$$

The total energy E_1, angular momentum ℓ_1, and the angle of the perigee can be used to determine the parameters c_1, ε_1, and δ_1. A brief impulse of the appropriate size and in the appropriate direction at angle ϕ_o can put the object into any orbit that crosses the original one (as shown below for bound orbits).



The values of the energy E_2 and angular momentum ℓ_2 after the impulse can be calculated. Those determine the values of c_2 and ε_2 for the new orbit. The condition $r_1(\phi_o) = r_2(\phi_o)$ determines the value of δ_2. This general case is somewhat complicated.

We will treat the special case of an impulse tangential to the orbit at the <u>perigee</u> (closest approach) of the initial <u>elliptical</u> orbit. This doesn't change the direction of the velocity (tangential) at the point of the impulse, so this point will be the perigee (or apogee) the new orbit. If we choose the axes so that $\delta_1 = 0$, then the impulse occurs at $\phi_o = 0$. For the new orbit, $\delta_2 = 0$, so there are two parameters to determine. Since the orbits must meet at $\phi_o = 0$:

$$\frac{c_1}{1+\varepsilon_1} = \frac{c_2}{1+\varepsilon_2}.$$

At perigee (or apogee), the size of the angular momentum is simply:

$$\ell = \mu r v,$$

because the velocity is tangential (perpendicular to $\vec{r}$). Suppose the relationship between the speed of the satellite just before and just after the impulse (rocket firing) is:

$$v_2 = \lambda v_1,$$

where λ is called the *thrust factor*. If $\lambda > 1$, then the thrust is forward and the final speed is larger. If $\lambda < 1$, then the thrust is backward and the final speed is smaller. The angular momentum changes by the same factor as the speed, so:

$$\ell_2 = \lambda \ell_1.$$

The parameter c is given by:

$$c = \frac{\ell^2}{\gamma\mu},$$

so the initial and final parameters are related by:

$$\boxed{c_2 = \lambda^2 c_1}.$$

Substituting this into the other relation for the orbits gives

$$\frac{c_1}{1+\varepsilon_1} = \frac{c_2}{1+\varepsilon_2} = \frac{\left(\lambda^2 c_1\right)}{1+\varepsilon_2},$$

$$1+\varepsilon_2 = \lambda^2\left(1+\varepsilon_1\right),$$

$$\boxed{\varepsilon_2 = \lambda^2\varepsilon_1 + \left(\lambda^2 - 1\right)}.$$

Case 1: $\lambda > 1$, so there is a forward thrust

This will result in an increased eccentricity, $\varepsilon_2 > \varepsilon_1$.

If λ is large enough, the final orbit will have $\varepsilon_2 \geq 1$ and it will be unbounded.

Case 2: $\lambda < 1$, so there is a backward thrust

This will result in a decreased eccentricity, $\varepsilon_2 < \varepsilon_1$.

If λ is small enough, the final orbit will have $\varepsilon_2 < 0$, which means that the perigee of the initial orbit is the apogee of the final orbit.

Example: (Prob. 8.34) A spacecraft starts in an orbit around the sun close to the earth (radius of $R_1 = 1$ AU, astronomical units). It is going to given a thrust to make it reach the orbit of Neptune (radius of about $R_2 = 30$ AU) at its apogee. (a) What is the required thrust factor? (b) How long will the spacecraft take to reach the apogee? (c) What second thrust factor will put the spacecraft into the same circular orbit as Neptune?

(a) The initial orbit is circular so $\varepsilon_1 = 0$ and $c_1 = R_1$. The final orbit must be at a distance of R_2 at the apogee of the second orbit, so:

$$R_2 = \frac{c_2}{1-\varepsilon_2}.$$

After the first thrust, $c_2 = \lambda^2 c_1$ and $\varepsilon_2 = \lambda^2\varepsilon_1 + \left(\lambda^2 - 1\right)$, so:

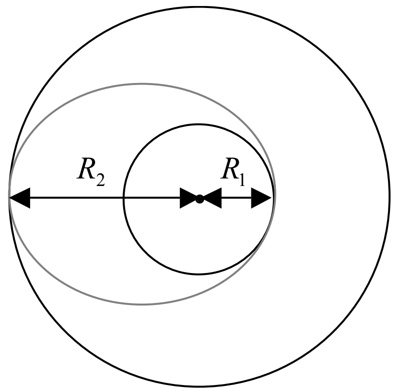
$$R_2 = \frac{\left(\lambda^2 c_1\right)}{1-\left[\lambda^2\varepsilon_1 + \left(\lambda^2 - 1\right)\right]} = \frac{\lambda^2 R_1}{2-\lambda^2},$$

$$\left(2-\lambda^2\right)R_2 = \lambda^2 R_1,$$

$$\lambda^2\left(R_1 + R_2\right) = 2R_2,$$

$$\boxed{\lambda = \sqrt{\frac{2R_2}{R_1 + R_2}} = \sqrt{\frac{2(30\text{ AU})}{1\text{ AU} + 30\text{ AU}}} = 1.39}.$$

(b) The orbits are shown below.



The semimajor axis of the transfer orbit (ellipse) is:

$$a = \frac{R_1 + R_2}{2} = \frac{1\text{ AU} + 30\text{ AU}}{2} = 15.5\text{ AU}.$$

The period of the initial orbit is $\tau_1 = 1$ yr, like the earth. According to Kepler's third law:

$$\tau^2 = \left(\frac{4\pi^2}{GM_s}\right) a^3,$$

so:

$$\frac{\tau_1^2}{\tau_2^2} = \frac{a_1^3}{a_2^3},$$

$$\boxed{\tau_2 = \tau_1 \left(\frac{a_2}{a_1}\right)^{3/2} = (1\text{ yr})\left(\frac{15.5\text{ AU}}{1\text{ AU}}\right)^{3/2} = 61\text{ yr}}.$$

(c) The second thrust occurs at the apogee of the transfer orbit (2), so continuity gives:

$$\frac{c_2}{1-\varepsilon_2} = \frac{c_3}{1-\varepsilon_3}.$$

Since the final orbit is circular, $\varepsilon_3 = 0$ and $c_3 = R_2$, so:

$$\frac{c_2}{1-\varepsilon_2} = R_2.$$

After the second thrust, $c_3 = R_2 = \lambda_2^2 c_2$, so $c_2 = R_2/\lambda_2^2$ and:

$$\frac{\left(R_2/\lambda_2^2\right)}{1-\varepsilon_2} = R_2,$$

$$\frac{1}{1-\varepsilon_2} = \lambda_2^2.$$

We already know that:

$$\varepsilon_2 = \lambda^2\varepsilon_1 + \left(\lambda^2 - 1\right) = \lambda^2 - 1$$

so:

$$\lambda_2^2 = \frac{1}{1-\left(\lambda^2-1\right)} = \frac{1}{2-\lambda^2}.$$

We also know that:

$$\lambda^2 = \frac{2R_2}{R_1 + R_2},$$

so:

$$\lambda_2^2 = \frac{1}{2-\left(\dfrac{2R_2}{R_1+R_2}\right)} = \frac{1}{\dfrac{2\left(R_1+R_2\right)-2R_2}{R_1+R_2}} = \frac{R_1+R_2}{2R_1}.$$

This is just the inverse of what we found in part (a) with the initial and final radii switched:

$$\boxed{\lambda_2 = \sqrt{\frac{R_1+R_2}{2R_1}} = \sqrt{\frac{1\ \text{AU} + 30\ \text{AU}}{2\left(1\ \text{AU}\right)}} = 3.94}.$$

Homework Hints

Problem 25 part b: google ‘root find’ and you’ll get an app that will find the roots of this; you only want the real ones.

Part c: in Vpython,

Start with some initial conditions

r = rmin

f = 0

Ball.pos = vector (r*cos(f), r*sin(f),0)

u = 1/r

u’ = 0 (since, right when r = rmin the radius momentarily stops shrinking, and so u stops growing)

in the loop

while f < 7*pi:

    u” = ….

    u’ = u ‘ + u”* delta_f

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u = u + u’ * delta_f
r = 1/u
f  = f + delta_f
Ball.pos = vector (r*cos(f), r*sin(f),0)
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