Wed. 11/14 Thurs. 11/15 Fri., 11/16	**8.7-8** Unbounded and Changing Orbits **9.1-.3** Noninertial Frames: Acceleration, Tides, Angular Velocity	**HW8b** (8.30, 8."36")

Last Time

Last time, we saw that

$$\frac{\ell^2}{\mu r^3} + F_r(r) = \mu\ddot{r}$$

Could be rephrased to show how the radial separation varies with the angular orientation (rather than variations with time)

$$\boxed{u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2} F(u)}$$

With the help of the definition

u = 1/r.

In the particular case of a force with the form

$$F(r) = -\frac{\gamma}{r^2} \quad \text{or} \quad F(u) = -\gamma u^2$$

where $\gamma = Gm_1 m_2$ for gravitation or $-\frac{1}{4\pi\varepsilon_o}|q_1 q_2|$ for electric,

then we have

$$u''(\phi) = -u(\phi) + \gamma\mu/\ell^2$$

And we could find a specific solution of the form

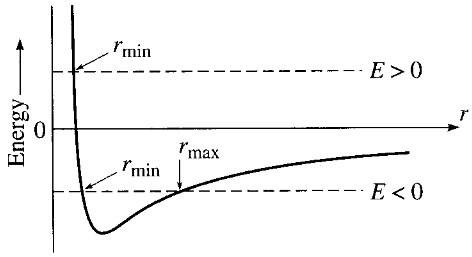
$$r(\phi) = \frac{1}{A\cos(\phi - \delta) + \gamma\mu/\ell^2}$$

To tidy things up, we defined $c \equiv \ell^2/\gamma\mu$ the *eccentricity* $\varepsilon \equiv A\ell^2/\gamma\mu = Ac$, allowing us to write it as

$$r(\phi) = \frac{c}{(1 + \varepsilon\cos\phi)}$$

Orbits, eccentricities, and energies. Qulaitatively, it's clear that if ε =0, the radius doesn't depend on the angle, so we've got a circle, if 0< ε <1, it varies between a maximum and minimum, so it's *some* closed shape, if ε =1, it blows up as ϕ approaches ϕ, and and if ε>1 it blows up as q approaches some intermediate angle.

Of course, energy considerations tell us something quite similar: there's a minimum energy for which the radius is held constant, higher energies allow the radius to vary between a minimum and maximum, and still higher allow it to stray all the way to infinity.



We saw that this connection between eccentricity and energy could be made explicit:

$$\boxed{E = \frac{\gamma^2 \mu}{2\ell^2}\left(\varepsilon^2 - 1\right)}$$

Finally, we sketched through the book's work to demonstrate that our "some closed shape" is really an ellipse with

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where:

$$a = \frac{c}{1-\varepsilon^2}, \quad b = \frac{c}{\sqrt{1-\varepsilon^2}}, \quad \text{and} \quad d = a\varepsilon = \frac{c\varepsilon}{1-\varepsilon^2}.$$

This Time

<u>Unbounded Orbits</u>:

The equation for Kepler orbits is:

$$r(\phi) = \frac{c}{(1+\varepsilon\cos\phi)}.$$

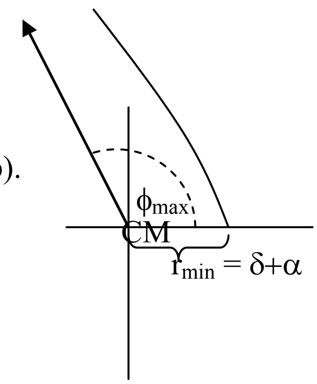
If <u>$\varepsilon = 1$</u>, $r(\phi) \to \infty$ as $\phi \to \pm\pi$ so the orbit is unbounded. However, there is no restriction on the angle. In the homework (problem 8.30), you'll put this in the form:

$$y^2 = c^2 - 2cx\,,$$

which is a <u>parabola</u> which opens toward the negative x direction; can also be written as $y^2 = 2c\left(\frac{c}{2} - x\right)$, so $r_{\min} = \frac{c}{1+\varepsilon} \underset{\varepsilon \to 1}{\to} \frac{c}{2}$.

This proceeds just as it did for rephrasing the equation as for an elipse:

$$r = \frac{c}{(1+\varepsilon\cos\phi)} \rightarrow \frac{c}{(1+\cos\phi)}$$
$$r(1+\cos\phi) = c$$
$$r + r\cos\phi = c$$
$$\sqrt{x^2+y^2} + x = c$$



If $\underline{\varepsilon > 1}$, then the radius blows up as the angle approaches an $\phi_{\max}$ given by:

$\cos\phi_{\max} = -1/\varepsilon$ (which makes the denominator blow up).

Of course, cosine is negative only for an angel greater than 90°.

The orbit will never go beyond $\pm\phi_{\max}$ because $r(\phi) \rightarrow \infty$ as $\phi \rightarrow \pm\phi_{\max}$.

The equation of orbit can be put in the form (also Prob. 8.30):

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1,$$

which is a hyperbola. You will determine the parameters α, β, and δ in the homework. (Note: α^2 and β^2 must be positive.)

Get them started on this.

$$r = \frac{c}{(1+\varepsilon\cos\phi)}$$
$$r(1+\varepsilon\cos\phi) = c$$
$$r + r\varepsilon\cos\phi = c$$
$$\sqrt{x^2+y^2} + x\varepsilon = c$$
$$x^2 + y^2 = (c - x\varepsilon)^2 = c^2 - 2cx\varepsilon + (x\varepsilon)^2$$
$$y^2 = c^2 + 2cx\varepsilon + x^2(\varepsilon^2 - 1)$$

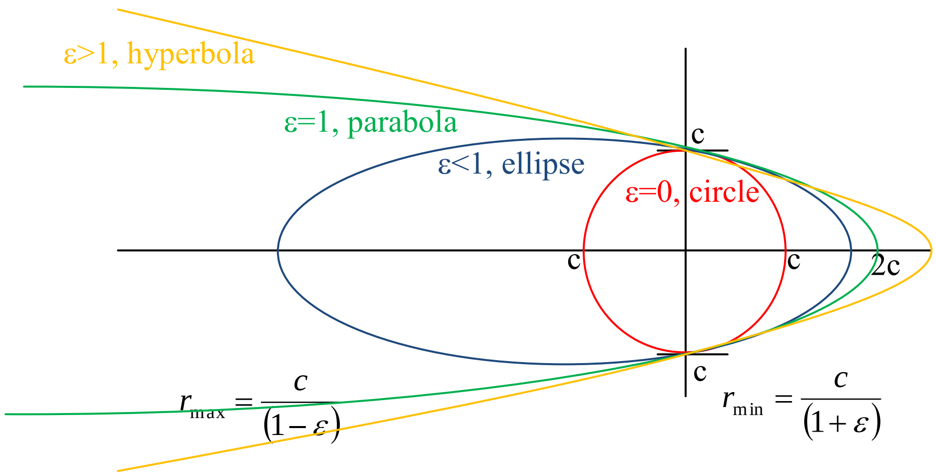
Complete the square for x…

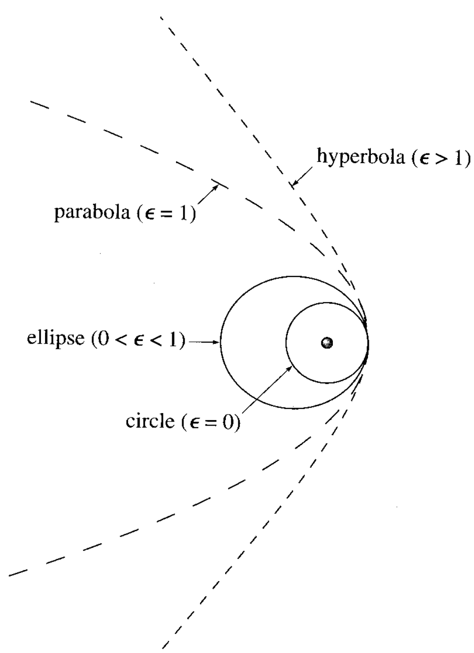
All of the "Kepler orbits" are described by $r(\phi)=\dfrac{c}{(1+\varepsilon\cos\phi)}$.

Eccentricity	Energy	Shape of Orbit
$\varepsilon=0$	$E<0$	circle
$0<\varepsilon<1$	$E<0$	ellipse
$\varepsilon=1$	$E=0$	parabola
$\varepsilon>1$	$E>0$	hyperbola

These are all "conic sections" – they are shapes that you get when you slice a cone different ways. The four different types of orbits are shown below ($\phi=0$ is horizontal to the right).

If we imagine holding c constant, then the shapes all have to go through the y axis (ϕ=π/2) at c.





In the equation $r(\phi)=\dfrac{c}{(1+\varepsilon\cos\phi)}$:

$$c=\frac{\ell^2}{\gamma\mu},\qquad \gamma=Gm_1m_2,\qquad \text{and}\qquad \mu=\frac{m_1m_2}{m_1+m_2}.$$

How can you find ℓ and ε?

1. Total angular momentum ($\ell=\left|\vec{r}\times\mu\dot{\vec{r}}\right|$) is easiest to find when $\vec{r}\perp\dot{\vec{r}}$.

   - Where does that occur?

     Point of closest approach (perigee) or farthest point (apogee) – the latter only exists for bounded orbits.

   - What is the value of ℓ at these points?

     <u>Only</u> at these points: $\ell=\mu rv$, where $v=\left|\dot{\vec{r}}\right|$ is the speed of one object relative to the other. At other points on the orbit, it would be much more difficult to determine the angle between $\vec{r}$ and $\dot{\vec{r}}$.

2. How can you find the eccentricity ε (after finding c)?

   Need to know the separation r at <u>any</u> angle ϕ. It is particularly easy to use the perigee or apogee because:

$$r_{\min}=c/(1+\varepsilon)\qquad\text{and}\qquad r_{\max}=c/(1-\varepsilon)$$

Example. Say we watch a comet zip in to the solar system, come to within 10^8km of the sun, and then zip out again, approaching an angle of $3\pi/4$ – it's following a hyperbolic path. How *fast* is the comet going when it's at the closes distance from the sun?

$$r_{\min}=\frac{c}{1+\varepsilon}\ \text{ where }\ c=\frac{l^2}{\mu\gamma}\ \text{ and }\ l=mv_{\max}r_{\min}\ \ \gamma=GM_sm\ \text{ for a Hyperbola, }\ \varepsilon=-\frac{1}{\cos\phi_{\max}}$$

We can use these four relations to figure out the maximum speed.

$$l=\mu v_{\max}r_{\min}\Rightarrow v_{\max}=\frac{l}{\mu r_{\min}}$$

$$c=\frac{l^2}{\mu\gamma}\Rightarrow l=\sqrt{c\mu\gamma}$$

$$r_{\min}=\frac{c}{1+\varepsilon},\varepsilon=-\frac{1}{\cos\phi_{\max}}\Rightarrow c=r_{\min}\left(1-\tfrac{1}{\cos\phi_{\max}}\right)$$

So

$$v_{\max} = \frac{\sqrt{r_{\min}\left(1 - \frac{1}{\cos\phi_{\max}}\right)\mu\gamma}}{\mu r_{\min}} = \sqrt{\left(1 - \frac{1}{\cos\phi_{\max}}\right)\frac{\gamma}{\mu r_{\min}}} = \sqrt{\left(1 - \frac{1}{\cos\phi_{\max}}\right)\frac{GM_s m}{\mu r_{\min}}}$$

Now, $\mu = \frac{M_s m}{M_s + m} \approx M_s$ if $M_s >> m$ (as is true for the sun and comet), so

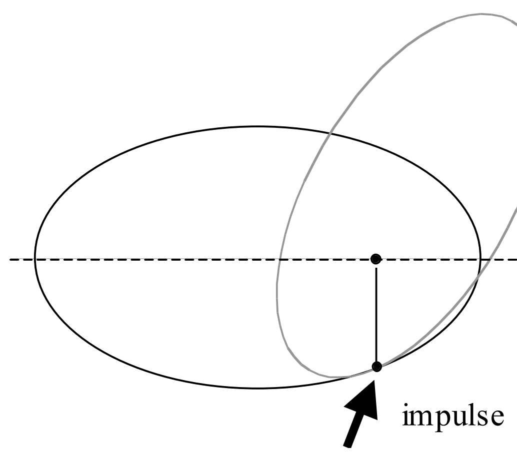
$$v_{\max} \approx \sqrt{\left(1 - \frac{1}{\cos\phi_{\max}}\right)\frac{GM_s m}{m r_{\min}}} = \sqrt{\left(1 - \frac{1}{\cos\phi_{\max}}\right)\frac{GM_s}{r_{\min}}}$$

<u>Changes of Orbit</u>:

The most general form of the equation of orbit is:

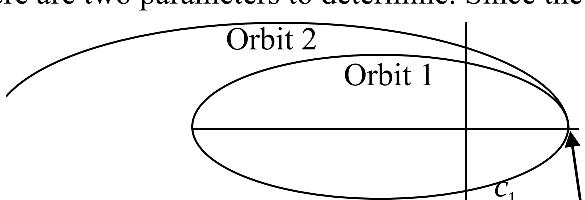
$$r_1(\phi) = \frac{c_1}{1 + \varepsilon_1 \cos(\phi - \delta_1)}.$$

The total energy E_1, angular momentum ℓ_1, and the angle of the perigee can be used to determine the parameters c_1, ε_1, and δ_1. A brief impulse of the appropriate size and in the appropriate direction at angle ϕ_o can put the object into any orbit that crosses the original one (as shown below for bound orbits).



The values of the energy E_2 and angular momentum ℓ_2 after the impulse can be calculated. Those determine the values of c_2 and ε_2 for the new orbit. The condition $r_1(\phi_o) = r_2(\phi_o)$ determines the value of δ_2. This general case is somewhat complicated.

We will treat the special case of an impulse tangential to the orbit at the <u>perigee</u> (closest approach) of the initial <u>elliptical</u> orbit. This doesn't change the direction of the velocity (tangential) at the point of the impulse, so this point will be the perigee (or apogee) the new orbit. If we choose the axes so that $\delta_1 = 0$, then the impulse occurs at $\phi_o = 0$. For the new orbit, $\delta_2 = 0$, so there are two parameters to determine. Since the orbits must meet at $\phi_o = 0$:



$$\frac{c_1}{1 + \varepsilon_1} = r_{\min} = \frac{c_2}{1 + \varepsilon_2}$$

At perigee (or apogee), the size of the angular momentum is simply:

$$\ell = \mu r v,$$

because the velocity is tangential (perpendicular to $\vec{r}$). Suppose the relationship between the speed of the satellite just before and just after the impulse (rocket firing) is:

$$v_2 = \lambda v_1,$$

where λ is called the *thrust factor*. If $\lambda > 1$, then the thrust is forward and the final speed is larger. If $\lambda < 1$, then the thrust is backward and the final speed is smaller. The angular momentum changes by the same factor as the speed, so:

$$\ell_2 = \lambda \ell_1.$$

The parameter c is given by:

$$c = \frac{\ell^2}{\gamma \mu},$$

so the initial and final parameters are related by:

$$\boxed{c_2 = \lambda^2 c_1}.$$

Substituting this into the other relation for the orbits gives

$$\frac{c_1}{1+\varepsilon_1} = \frac{c_2}{1+\varepsilon_2} = \frac{(\lambda^2 c_1)}{1+\varepsilon_2},$$

$$1+\varepsilon_2 = \lambda^2 (1+\varepsilon_1),$$

$$\boxed{\varepsilon_2 = \lambda^2 \varepsilon_1 + (\lambda^2 - 1) = (\lambda^2 + 1)\varepsilon_1 - 1}.$$

So, *given* the parameters of the initial orbit, for a given thrust factor, you can determine the parameters of the new orbit.

General Case 1: $\lambda > 1$, so there is a forward thrust

This will result in an increased eccentricity, $\varepsilon_2 > \varepsilon_1$.

If λ is large enough, the final orbit will have $\varepsilon_2 \geq 1$ and it will be unbounded.

General Case 2: $\lambda < 1$, so there is a backward thrust

This will result in a decreased eccentricity, $\varepsilon_2 < \varepsilon_1$.

If λ is small enough, the final orbit will have $\varepsilon_2 < 0$, which means that the perigee of the initial orbit is the apogee of the final orbit.

Jump to a circle:

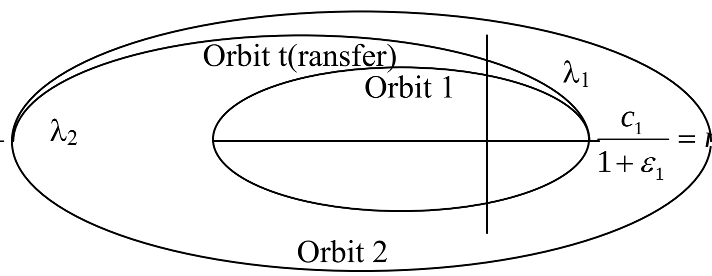
$$\varepsilon_2 = 0 = (\lambda^2 + 1)\varepsilon_1 - 1 \Rightarrow \lambda = \pm\sqrt{\frac{1}{\varepsilon_1} - 1}$$

Jump to a parabola:

$$\varepsilon_2 = 1 = (\lambda^2 + 1)\varepsilon_1 - 1 \Rightarrow \lambda = \pm\sqrt{\frac{2}{\varepsilon_1} - 1}$$

This works for getting you between any two orbits that have the same perigee (point of closest approach). What about jumping between two orbits that *don't* have the same perigee, in fact, don't share any point at all (but are about the same center of mass, say, the sun)? You can do that with *two jumps* between *three* orbits. We'll call the middle orbit a "transfer" orbit. For example, say you make this transition:

$$\frac{c_2}{1-\varepsilon_2} = r_{\max} = \frac{c_t}{1-\varepsilon_t}$$



$$\frac{c_1}{1+\varepsilon_1} = r_{\min} = \frac{c_t}{1+\varepsilon_t}$$

Say you *know* what orbit you're starting with and what orbit you want to end in, so the question really is what boosts you need to get between the two, λ_1 and λ_2.

As intermediate steps, we'll need to find the parameters of the transfer orbit.

$$c_2\frac{1-\varepsilon_t}{1-\varepsilon_2} = c_t = c_1\frac{1+\varepsilon_t}{1+\varepsilon_1} \Rightarrow \varepsilon_t = \frac{\frac{c_2}{1-\varepsilon_2} - \frac{c_1}{1-\varepsilon_1}}{\frac{c_2}{1-\varepsilon_2} + \frac{c_1}{1-\varepsilon_1}} = \frac{r_{\max.2} - r_{\min.1}}{r_{\max.2} + r_{\min.1}}$$

With that in hand, we can find the scale factor for the transfer orbit, c_t,

$$\frac{c_1}{1+\varepsilon_1} = \frac{c_t}{1+\varepsilon_t} \Rightarrow c_t = \frac{c_1}{1+\varepsilon_1}(1+\varepsilon_t) = \frac{c_1}{1+\varepsilon_1}\left(1 + \frac{r_{\max.2} - r_{\min.1}}{r_{\max.2} + r_{\min.1}}\right) = 2\left(\frac{r_{\min.1} r_{\max.2}}{r_{\max.2} + r_{\min.1}}\right)$$

Then using these two expressions, we can find the two necessary boost

$$c_t\lambda_2^2 = c_2 \Rightarrow \lambda_2 = \sqrt{\frac{c_2}{c_t}}$$

And $\lambda_1 = \sqrt{\frac{c_t}{c_1}}$

Example: (Prob. 8.34) A spacecraft starts in a circular orbit around the sun close to the earth (radius of $R_1 = 1\text{ AU}$, astronomical units). It is going to be given a thrust to make it reach the circular orbit of Neptune (radius of about $R_2 = 30\text{ AU}$) at the apogee of its transfer orbit (an ellipse). (a) What is the required thrust factor? (b) How long will the spacecraft take to reach the apogee? (c) What second thrust factor will put the spacecraft into the same circular orbit as Neptune?

(a) The initial orbit is essentially circular so $\varepsilon_1 = 0$ and $c_1 = R_1$. The final orbit must be at a distance of R_2 at the apogee of the second orbit, so:

$$R_2 = \frac{c_2}{1-\varepsilon_2}.$$

After the first thrust, $c_2 = \lambda^2 c_1$ and $\varepsilon_2 = \lambda^2\varepsilon_1 + (\lambda^2 - 1)$, so:

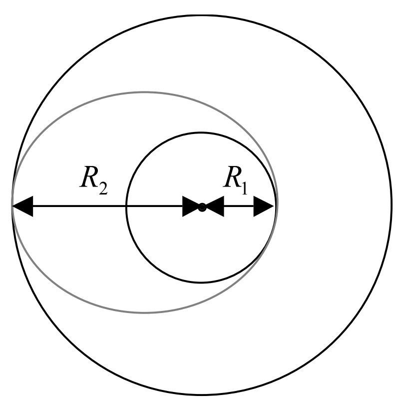
$$R_2 = \frac{(\lambda^2 c_1)}{1-[\lambda^2\varepsilon_1 + (\lambda^2-1)]} = \frac{\lambda^2 R_1}{2-\lambda^2},$$

$$(2-\lambda^2)R_2 = \lambda^2 R_1,$$

$$\lambda^2(R_1 + R_2) = 2R_2,$$

$$\boxed{\lambda = \sqrt{\frac{2R_2}{R_1+R_2}} = \sqrt{\frac{2(30\text{ AU})}{1\text{ AU} + 30\text{ AU}}} = 1.39}.$$

(b) The orbits are shown below.



The semimajor axis of the transfer orbit (ellipse) is:

$$a = \frac{R_1+R_2}{2} = \frac{1\text{ AU} + 30\text{ AU}}{2} = 15.5\text{ AU}.$$

The period of the initial orbit is $\tau_1 = 1$ yr, like the earth. According to Kepler's third law:

$$\tau^2 = \left(\frac{4\pi^2}{GM_s}\right)a^3,$$

so:

$$\frac{\tau_1^2}{\tau_2^2} = \frac{a_1^3}{a_2^3},$$

$$\boxed{\tau_2 = \tau_1\left(\frac{a_2}{a_1}\right)^{3/2} = (1\ \text{yr})\left(\frac{15.5\ \text{AU}}{1\ \text{AU}}\right)^{3/2} = 61\ \text{yr}}.$$

Then the in that second, transfer orbit is just half the period, or 30.5 yrs.

(c) The second thrust occurs at the apogee of the transfer orbit (2), so continuity gives:

$$\frac{c_2}{1-\varepsilon_2} = \frac{c_3}{1-\varepsilon_3}.$$

Since the final orbit is circular, $\varepsilon_3 = 0$ and $c_3 = R_2$, so:

$$\frac{c_2}{1-\varepsilon_2} = R_2.$$

After the second thrust, $c_3 = R_2 = \lambda_2^2 c_2$, so $c_2 = R_2/\lambda_2^2$ and:

$$\frac{\left(R_2/\lambda_2^2\right)}{1-\varepsilon_2} = R_2,$$

$$\frac{1}{1-\varepsilon_2} = \lambda_2^2.$$

We already know that:

$$\varepsilon_2 = \lambda^2\varepsilon_1 + \left(\lambda^2 - 1\right) = \lambda^2 - 1$$

so:

$$\lambda_2^2 = \frac{1}{1-\left(\lambda^2-1\right)} = \frac{1}{2-\lambda^2}.$$

We also know that:

$$\lambda^2 = \frac{2R_2}{R_1+R_2},$$

so:

$$\lambda_2^2 = \frac{1}{2-\left(\dfrac{2R_2}{R_1+R_2}\right)} = \frac{1}{\dfrac{2(R_1+R_2)-2R_2}{R_1+R_2}} = \frac{R_1+R_2}{2R_1}.$$

This is just the inverse of what we found in part (a) with the initial and final radii switched:

$$\boxed{\lambda_2 = \sqrt{\frac{R_1+R_2}{2R_1}} = \sqrt{\frac{1\ \text{AU}+30\ \text{AU}}{2(1\ \text{AU})}} = 3.94}.$$