Mon., 11/12 Tues., 11/13 Wed., 11/14 Thurs. 11/15 Fri., 11/16	**8.5-.6** Orbits **8.7-8** Unbounded and Changing Orbits **9.1-.3** Noninertial Frames: Acceleration, Tides, Angular Velocity	**HW8a** (8.8-.25) **HW8b** (8.30, 8."36")

Last Time:

$$E_{system} = \tfrac{1}{2} M\dot{\vec{R}}_{cm}^2 + \tfrac{1}{2}\mu\dot{\vec{r}}^2_{12} + U_{1,2}$$

$$M \equiv (m_1 + m_2)$$

$$\mu \equiv \frac{m_1 m_2}{(m_1 + m_2)}$$

In Cartesian: $\tfrac{1}{2}\mu\dot{\vec{r}}^2 = \tfrac{1}{2}\mu(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

In Polar/Cylindrical: $\tfrac{1}{2}\mu\dot{\vec{r}}^2 = \tfrac{1}{2}\mu(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$

In Spherical: $\tfrac{1}{2}\mu\dot{\vec{r}}^2 = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2)$

$$\vec{L}_c = \vec{r}_1 \times m_1\dot{\vec{r}}_1 + \vec{r}_2 \times m_2\dot{\vec{r}}_2 = \vec{r}_{12} \times \mu\dot{\vec{r}}_{12},$$

(dropping the subscript)

$$\vec{r} = r\hat{r} \quad \text{and} \quad \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}.$$

Substituting these in gives:

$$\vec{L} = (r\hat{r}) \times \mu(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}) = (\mu r^2\dot{\phi}\,\hat{z}),$$

Note: at any given moment, these two vectors are in a plane, we'll *define* the direction that's perpendicular to the two as the z direction.

With no external torques applied to the system, this angular momentum vector must be constant in both magnitude, $\ell = \mu r^2\dot{\phi}$, and direction.

$$\boxed{\dot{\phi} = \frac{\ell}{\mu r^2}}$$

At any given moment, these two vectors, or in terms of cylindrical polar coordinates (since this is a 2-D problem):

$$E_{system} = \tfrac{1}{2} M\dot{\vec{R}}_{cm}^2 + \tfrac{1}{2}\mu(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \cancel{\dot{z}^2}) + U_{1,2}$$

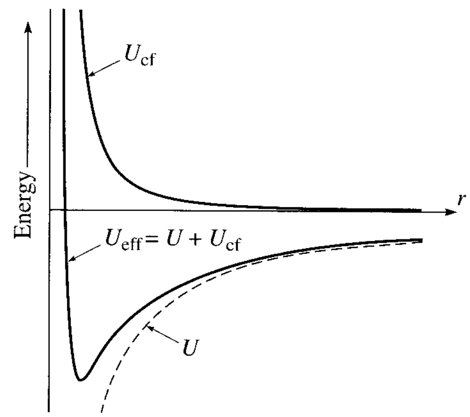
$$E_{system} = \tfrac{1}{2} M\dot{\vec{R}}_{cm}^2 + \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + 0 + U(r) = \tfrac{1}{2} M\dot{\vec{R}}_{cm}^2 + \tfrac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

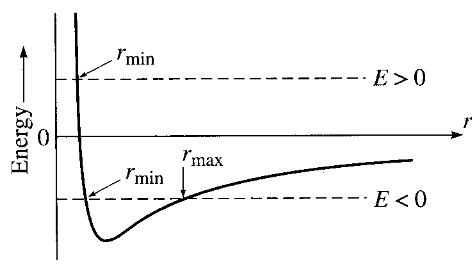
Where

$$U_{eff}(r) \equiv \frac{\ell^2}{2\mu r^2} + U(r)$$

Looking just at the energy associated with relative motion & position, i.e., looking in the center-of-mass frame,

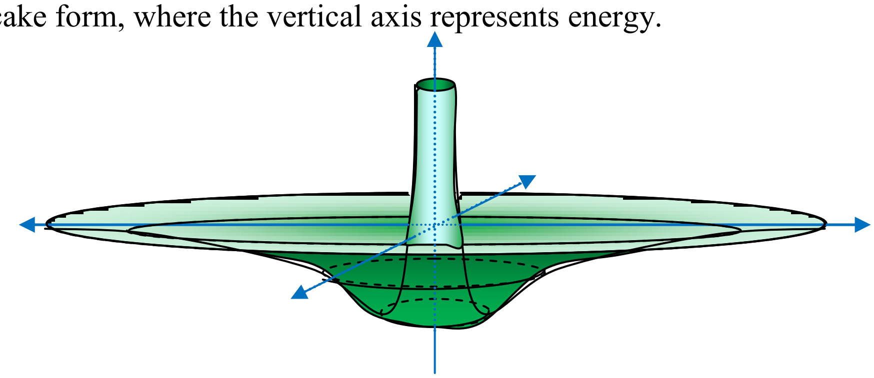
$$E = \tfrac{1}{2}\mu \dot{r}^2 + U_{eff}(r)$$

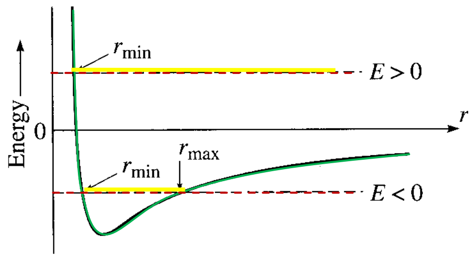


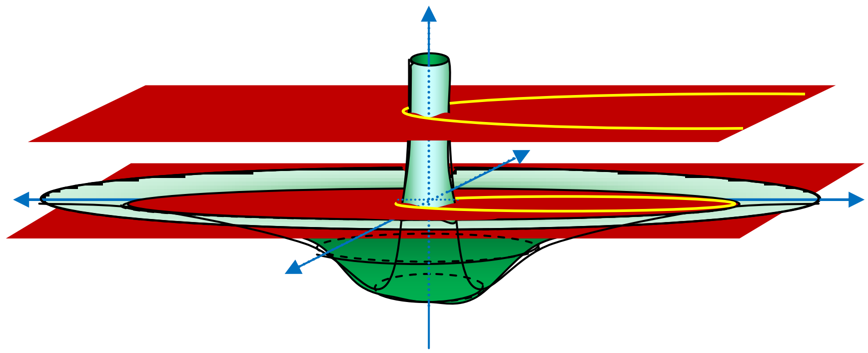


- If $E < 0$ and $E > (U_{eff})_{min}$, there are two turning points, r_{min} and r_{max}, so the motion is *bounded.*
- If the energy is equal to the minimum value of U_{eff}, then $\dot{r} = 0$ and the radius is constant. That means the orbit is circular.
- If $E > 0$, there is just one turning point, r_{min}, so the object can move off to infinity. The orbit is *unbounded* in this case.

Mind you, we're only plotting the energy vs. r. It may be useful to visualize the potential well in the full x-y plane (conveniently, that's where all the action is). It looks like a flaired bunt cake form, where the vertical axis represents energy.







A comet with E<0 would be bound – having minimum and maximum radii where the constant-energy plane hits the outer and inner wall of the potential form.

A comet with E>0 would be unbound – having a minimum, but no maximum radius.

Using the Lagrangian approach, we can say that

$$\mathcal{L}_r = T_r - U_{eff}$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$-\frac{\partial}{\partial r}U_{eff} = \frac{d}{dt}\mu\dot{r}$$

$$-\frac{\partial}{\partial r}U_{eff} = \mu\ddot{r}$$

Or,

$$-\frac{\partial}{\partial r}\left(\frac{\ell^2}{2\mu r^2} + U(r)\right) = \mu\ddot{r}$$

$$\frac{\ell^2}{\mu r^3} - \frac{\partial}{\partial r}U(r) = \mu\ddot{r}$$

$$\frac{\ell^2}{\mu r^3} + F_r(r) = \mu\ddot{r}$$

Of course, if we're talking about a central force, then the radial component is the *only* component, so I might as well drop the subscript.

$$\frac{\ell^2}{\mu r^3} + F(r) = \mu\ddot{r}$$

Now, that's as general as I can keep it. If you know an expression for the force, then plug it in and you've got an expression for finding how *r* varies with *time*. Maybe you can solve this expression for an analytical function *r(t)*, or maybe you just use this in an Euler-Cromer computational approach.

Of course, once you've got either an expression or a value for *r,* you can use

$$\dot{\phi} = \frac{\ell}{\mu r^2}$$

to update to find the corresponding angle.

Demo: orbit (for a crazy central force, F= - kr$^{-5/2}$)

This time

Equation of Orbit:

When we think of an orbit we think of the geometric pattern that a planet or whatever traces through space – not so much where is the planet at what *time* (which the above equation can help us find), but what's equation of the whole path. So we want to rephrase how *r* and how ϕ varies with time as how *r* varies with ϕ.

The first step is:

$$\frac{dr}{dt} = \frac{d\phi}{dt}\frac{dr}{d\phi} = \dot{\phi}\frac{dr}{d\phi}$$

If this seems like mathematical magic, think of this example – say a bug is walking radially outward on a spinning record. The record goes around once every 2 seconds and the bug walks out 3 cm each time it goes around, then at what rate is the bug walking out?

$$\frac{1rev}{2s}\frac{3cm}{1rev} = \frac{3cm}{2s}$$

That's the logic we're using above.

Okay, so it makes sense, but is it any *use,* aren't we just trading one time derivative for another? Yes, but… since angular momentum is constant, we can rephrase the angular time derivative as

$$\dot{\phi} = \frac{\ell}{\mu r^2}$$

So,

$$\frac{dr}{dt} = \dot{\phi}\frac{dr}{d\phi} = \frac{\ell}{\mu r^2}\frac{dr}{d\phi}$$

Which may be *uglier* than what we started with, but it no longer has explicit time dependence.

So we can now rephrase

$$\mu\ddot{r} = F + \frac{\ell^2}{\mu r^3}$$

$$\mu\frac{d}{dt}\left(\frac{d}{dt}r\right) = F + \frac{\ell^2}{\mu r^3}$$

$$\mu\dot{\phi}\frac{d}{d\phi}\left(\dot{\phi}\frac{dr}{d\phi}\right) = F + \frac{\ell^2}{\mu r^3}$$

$$\mu\frac{\ell}{\mu r^2}\frac{d}{d\phi}\left(\frac{\ell}{\mu r^2}\frac{dr}{d\phi}\right) = F + \frac{\ell^2}{\mu r^3}$$

$$\frac{d}{d\phi}\left(\frac{1}{r^2}\frac{dr}{d\phi}\right) = r^2\frac{\mu F}{\ell^2} + \frac{1}{r}$$

Now, if you note that

$$\frac{1}{r^2}\frac{dr}{d\phi} = -\frac{d}{d\phi}\left(\frac{1}{r}\right)$$

It's appealing to define u = 1/r. That lets us write all this as

$$-\frac{d}{d\phi}\left(\frac{du}{d\phi}\right) = \frac{\mu F}{u^2\ell^2} + u$$

$$u'' = -\left(\frac{\mu F}{u^2\ell^2} + u\right)$$

$$\boxed{u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}F(u)}.$$

(indicating upon what things depend)

This is the differential equation for *<u>any</u>* central force, if $\ell \neq 0$ (can't divide by it in that case, then again, ϕ=constant, so there's no way to parameterize u in terms of ϕ).

The equation of orbit is solved for $u(\phi)$ which is inverted to get $r(\phi)$. The solution can be transformed to Cartesian coordinates (*x* and *y*) to see what kind of function it is (we're more familiar with ellipses, parabolas, and hyperbolas in that form).

Example: spring force.

Say $F = -k(r - r_o) = -\frac{k}{u} + kr_o$

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}\left(-\frac{k}{u} + kr_o\right)$$

$$u''(\phi) = -u(\phi) + \frac{\mu k}{\ell^2 u(\phi)^3} - \frac{\mu k r_o}{\ell^2 u(\phi)^2}$$

<u>Kepler Orbits</u>:

It so happens that the math is simpler for our favorite force, ~1/r^2, as in

$F_{elect} = -\frac{1}{4\pi\varepsilon_o}\frac{|q_1 q_2|}{r^2}$ when charges of opposite sign interact or $F_{grav} = -G\frac{m_1 m_2}{r^2}$

To remain somewhat general, we'll just say we're concerned with a force of the form

$$F(r) = -\frac{\gamma}{r^2} \quad \text{or} \quad F(u) = -\gamma u^2,$$

where $\gamma = Gm_1 m_2$ for gravitation or $-\frac{1}{4\pi\varepsilon_o}|q_1 q_2|$ for electric.

Now, the equation of orbit is:

$$u''(\phi) = -u(\phi) + \gamma\mu/\ell^2.$$

Or

$$u''(\phi) + u(\phi) = \gamma\mu/\ell^2$$

This has the *exact* same form as the equation for a simple harmonic oscillator with an offset.

$$u(\phi) = A\cos(\phi - \delta) + \gamma\mu/\ell^2$$

Solves it.

You can deduce that by saying we could guess a *homogeneous* solution (what the solution would be if the right hand side were 0), and then add it to a particular solution (one that gives us the right-hand-side). The homogeneous solution is the familiar

$u_h(\phi) = A\cos(\phi - \delta)$

Then adding on the constant as the particular solution

$u_p(\phi) = \gamma\mu/\ell^2$

So

$u(\phi) = u_h(\phi) + u_p(\phi)$
$u(\phi) = A\cos(\phi - \delta) + \gamma\mu/\ell^2$

Let's choose to define our coordinate system oriented such that $u(0) = A + \gamma\mu/\ell^2$

That's u's *maximal* value, or 1/u = r's *minimal* value.

$u(0) = A\cos(-\delta) + \gamma\mu/\ell^2 = A + \gamma\mu/\ell^2$ meaning that $\delta = 0$.

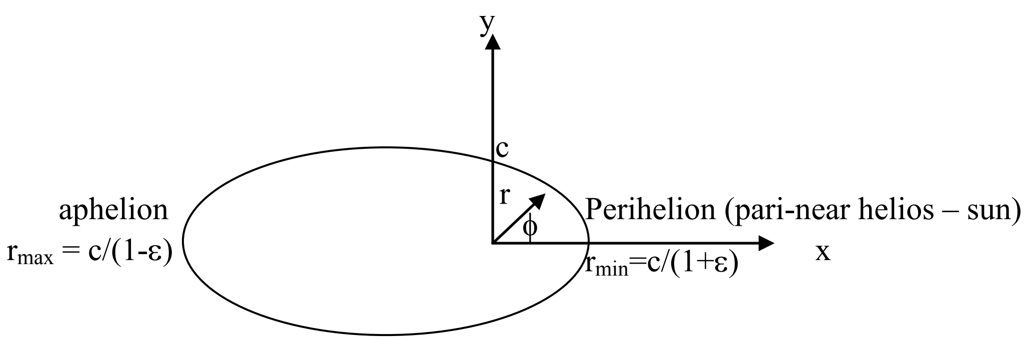
$u(\phi) = A\cos(\phi) + \gamma\mu/\ell^2 = \frac{\gamma\mu}{\ell^2}\left(\frac{\ell^2 A}{\gamma\mu}\cos(\phi) + 1\right)$

If we define $c \equiv \ell^2/\gamma\mu$ the *eccentricity* $\varepsilon \equiv A\ell^2/\gamma\mu = Ac$ (a dimensionless constant), and then the solution is:

$$u(\phi) = \frac{1}{r(\phi)} = \frac{(1 + \varepsilon\cos\phi)}{c},$$

so:

$$\boxed{r(\phi) = \frac{c}{(1 + \varepsilon\cos\phi)}}.$$



This is the general solution for the relative position in the gravitational, two-body problem. There are two coefficients which depend upon the particulars of the orbit – c can be phrased as depending upon the angular momentum (or vice versa) while ε also depends upon the 'amplitude', A.

We traded two 2nd-order differential equations (with respect to time) for one 2nd-order equation in terms of angle which gave really two distinct constants (taking *l* as a given): A and δ, but we were free to define our coordinate system such that $\delta = 0$.

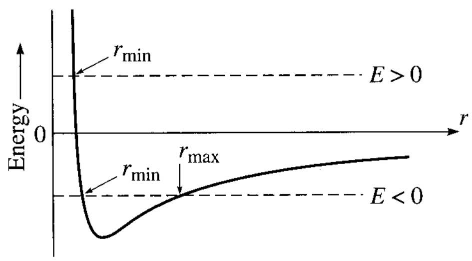
The *type* of curve that it describes depends on the value of *ε*. The two cases to consider are:

1. $\underline{\varepsilon < 1}$: The denominator of $r(\phi)$ never vanishes, so $r(\phi)$ is *bounded*. It has a finite maximum radial distance.
2. $\underline{\varepsilon \geq 1}$: The denominator of $r(\phi)$ vanishes (the whole thing blows up) for some angle, so $r(\phi)$ is *unbounded*. The radial distance approaches infinity as ϕ approaches this angle.

Energy – eccentricity relation

This is the same distinction that we saw between the cases $E < 0$ and $E \geq 0$ when we discussed the effective potential. Indeed, we can relate the two:

Returning to our energy plot,



at its distance of closest approach r_{min}, the orbiting object's energy is equal to the effective potential energy U_{eff} for that separation. That means:

$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2} = \frac{1}{r_{min}}\left(\frac{\ell^2}{2\mu r_{min}} - \gamma\right).$$

The minimum radial distance is when cosine = 1, so:

$$r_{min} = \frac{c}{1+\varepsilon} = \frac{\ell^2}{\gamma\mu(1+\varepsilon)},$$

so:

$$E = \frac{\gamma\mu(1+\varepsilon)}{\ell^2}\left[\frac{\gamma(1+\varepsilon)}{2} - \gamma\right] = \frac{\gamma^2\mu}{2\ell^2}\left[(1+\varepsilon)^2 - 2(1+\varepsilon)\right],$$

$$\boxed{E = \frac{\gamma^2\mu}{2\ell^2}\left(\varepsilon^2 - 1\right)}.$$

The relationship between energy and eccentricity is $E < 0$ if $\varepsilon < 1$ (bound) and $E \geq 0$ if $\varepsilon \geq 1$ (unbound).

<u>Bounded Orbits</u>: Kepler's 1st law

If $\varepsilon < 1$, the denominator of the following never vanishes:

$$r(\phi) = \frac{c}{(1+\varepsilon\cos\phi)},$$

so the minimum distance (*perihelion* around sun or *perigee* in general) is:

$$r_{\min} = r(\phi = 0) = \frac{c}{1+\varepsilon},$$

and the maximum distance (*aphelion* around the sun or *apogee* in general) is:

$$r_{\max} = r(\phi = \pi) = \frac{c}{1-\varepsilon}.$$

Example (Prob. 8.16) We want to show Kepler's first law – bounded orbits are ellipses. First, we need to know what equation describes an ellipse.

For a *circle* centered on the origin, $\left(\frac{x}{R}\right)^2 + \left(\frac{y}{R}\right)^2 = 1$

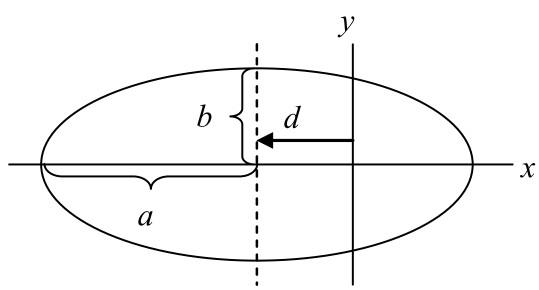
For an *elipse,* An ellipse centered on the origin $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. Essentially the two axes are scaled differently.

Obviously, when x = 0, $\frac{y^2}{b^2} = 1$, $y = \pm b$ are the two extreme values of y; similarly, when y = 0, $\frac{x^2}{a^2} = 1$, $x = \pm a$ are the two extreme values of x.

If a > b, then we'd say *a* is the "semi-major" axis (semi – half, major – larger) and *b* is the "semi-minor" axis.

An ellipse shifted in the *x* direction (see the diagram below), is described by:

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Now, when y = 0, this gives

$$\frac{(x+d)^2}{a^2} = 1$$

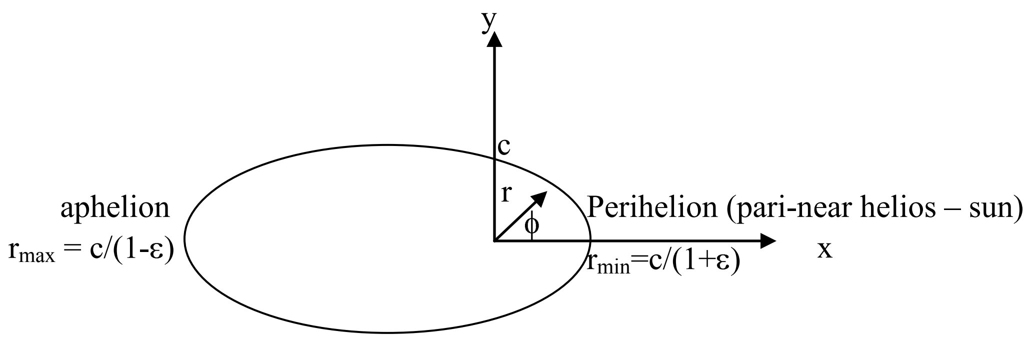
$$(x+d)^2 = a^2$$

$$x + d = \pm a$$

$$x = \pm a - d$$

So the full range of *x* is still 2*a* wide, and *a* is still the "Semimajor axis."

Now, let's see if we can massage our orbital equation into this form.



Multiplying both sides of the equation of orbit by $(1+\varepsilon\cos\phi)$ gives:

$$r(1+\varepsilon\cos\phi) = c$$

$$r + \varepsilon(r\cos\phi) = c,$$

$$r + \varepsilon x = c,$$

,

since $x = r\cos\phi$.

$$r = c - \varepsilon x$$

Of course, $r^2 = x^2 + y^2$, so let's square both sides and then replace r^2.

Square both sides: $x^2 + y^2 = r^2 = c^2 - 2c\varepsilon x + \varepsilon^2 x^2$,

$$(1-\varepsilon^2)x^2 + 2c\varepsilon x + y^2 = c^2.$$

Divide both sides by $(1-\varepsilon^2)$ and define $d = c\varepsilon/(1-\varepsilon^2)$ to get:

$$(x^2 + 2d\cdot x) + \frac{y^2}{1-\varepsilon^2} = \frac{c^2}{1-\varepsilon^2},$$

and add d^2 to both sides to complete the square (in the brackets):

$$(x^2 + 2d\cdot x + d^2) + \frac{y^2}{1-\varepsilon^2} = \frac{c^2}{1-\varepsilon^2} + d^2 = \frac{c^2}{1-\varepsilon^2} + \frac{c^2\varepsilon^2}{(1-\varepsilon^2)^2} = \frac{c^2}{(1-\varepsilon^2)^2}.$$

Define $a = c/\left(1-\varepsilon^2\right)$ and divide by a^2 to get:

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{a^2\left(1-\varepsilon^2\right)} = 1.$$

If we define $b = a\sqrt{1-\varepsilon^2}$, then:

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where:

$$a = \frac{c}{1-\varepsilon^2}, \quad b = \frac{c}{\sqrt{1-\varepsilon^2}}, \quad \text{and} \quad d = a\varepsilon = \frac{c\varepsilon}{1-\varepsilon^2}.$$

Again, the distance *a* is called the *semimajor axis* and *b* is called the *semiminor axis*. The ratio of the lengths of the axes is:

$$\frac{b}{a} = \sqrt{1-\varepsilon^2}.$$

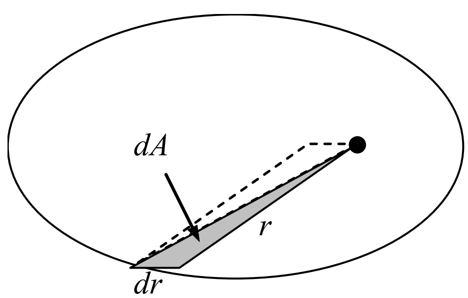
For bounded orbits, the parameter ε is called the *eccentricity*, as you can now see, it's a measure of how mismatched the two axes are. If $\varepsilon = 0$, then a=b and orbit is just as tall as it is wide; it's a circle. As $\varepsilon \to 1$, the ellipse becomes thin and elongated. The two-body system's center of mass is at one focus of the ellipse (for most practical purposes, that's the sun).

Kepler's Laws

We have already shown (Ch. 3) Kepler's second law – the rate at which a line from the sun a planet (or comet) sweeps out area *A* at a constant rate.

From geometry (which we reasoned through in Chapter 3), the area of the differentially thing triangle illustrated below is half that of the parallelogram shown, which itself is simply the cross of its two sides.

$$dA = \tfrac{1}{2}\left|\vec{r} \times d\vec{r}\right|.$$



So, the rate with which the area is being swept through is

$$\frac{dA}{dt} = \tfrac{1}{2}\left|\vec{r}\times\vec{v}\right| = \tfrac{1}{2}\frac{\left|\vec{r}\times\mu\vec{v}\right|}{\mu} = \frac{\left|\vec{r}\times\vec{p}\right|}{2\mu} = \frac{\left|\vec{L}_{c.m.}\right|}{2\mu} = \frac{\ell}{2\mu}$$

Finally, we would like to derive Kepler's third law, which says that the square of the period of an elliptical orbit is proportional to the cube of the semimajor axis.

The area of an ellipse is $A = \pi ab$ (it is a "circle with one dimension stretched"). If the period of an orbit is τ, then the rate of sweeping out area is:

$$\frac{(\pi ab)}{\tau} = \frac{\ell}{2\mu},$$

so:

$$\tau = \frac{2\pi ab\mu}{\ell}.$$

From earlier, we know that $b = a\sqrt{1-\varepsilon^2}$. Squaring both sides and substituting in for b gives:

$$\tau^2 = \frac{4\pi^2 a^4 (1-\varepsilon^2)\mu^2}{\ell^2},$$

but $c = a(1-\varepsilon^2)$ and $c = \ell^2/\gamma\mu$ so:

$$\tau^2 = \left(\frac{4\pi^2 c\mu^2}{\ell^2}\right)a^3 = \left(\frac{4\pi^2\mu}{\gamma}\right)a^3.$$

Now, this much is true for an orbit under the influence of any $-1/r^2$ force.

Now we'll get specific to a gravitational one with a planet around a much more massive star. Since the mass of the sun $m_2 = M_s$ is large compared to the mass of orbiting bodies, we can approximate the total mass for the system is $M = m_1 + m_2 \approx M_s$ and the reduced mass is $\mu = m_1 m_2/M \approx m_1$. That means that $\gamma = Gm_1m_2 \approx G\mu M$ and (dropping the approximation symbol because this is a very good approximation):

$$\tau^2 \approx \left(\frac{4\pi^2}{GM_s}\right)a^3.$$