Fri., 11/9	**8.3-.4** Equations of Motion for 2-Body Central Force	
Mon., 11/12	**8.5-.6** Orbits	
Tues., 11/13		**HW8a** (8.8-.25)
Wed., 11/14	**8.7-8** Unbounded and Changing Orbits	
Thurs. 11/15		**HW8b** (8.30, 8."36")
Fri., 11/16	**9.1-.3** Noninertial Frames: Acceleration, Tides, Angular Velocity	

Before we were so rudely interrupted by an exam, we'd begun looking at 2-body central force problems.

I started by reminding you that it's often convenient to split up the energy for a compound system into the energy associated with the translation of the center of mass and that associated with motion relative to the center of mass.

If $\vec{R}_{cm} \equiv \frac{\sum m_i \vec{r}_i}{\sum m_i}$, then

$$E_{system} = \sum_i \left(T_i + \sum_{j<i} U_{i,j} \right) = T_{CM} + \sum_i \left(T_{i-cm} + \sum_{j<i} U_{i,j} \right) = \tfrac{1}{2} M_{total} \dot{R}_{cm}^2 + \sum_i \left(\tfrac{1}{2} m_i \dot{r}_{i-cm}^2 + \sum_{j<i} U_{i,j} \right)$$

where $\vec{r}_{i-cm} = \vec{r}_i - \vec{R}_{cm}$

For example, when a ball goes flying through the air, you can track the center-of-mass motion, but, on the atomic scale, there's also a lot of jiggling and bonds that hold it together, or when you look at a distant star system, may be all that you can resolve is the 'net' motion of the system, the kinetic energy of the center of mass, but maybe there are a few stars and planets that make up that system and there's energy associated with their gravitational interactions and their motions relative to the center of mass.

Of course, the simplest 'compound' system is one of just two particles. In that case, our sums reduce to

$$E_{system} = \tfrac{1}{2} m_1 \dot{r}_1^2 + \tfrac{1}{2} m_2 \dot{r}_2^2 + U_{1,2}$$

In that case, you get a particularly nice expression

$$E_{system} = \tfrac{1}{2} (m_1 + m_2) \dot{R}_{cm}^2 + \tfrac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} \dot{r}_{12}^2 + U_{1,2}$$

So, if we define

$$M \equiv (m_1 + m_2)$$

$$\mu \equiv \frac{m_1 m_2}{(m_1 + m_2)}$$

We can write the system's energy in short-hand as

$$E_{system} = \tfrac{1}{2} M\dot{R}_{cm}^2 + \tfrac{1}{2}\mu\dot{r}^2{}_{12} + U_{1,2}$$

This is a particularly convenient way to express the kinetic energy because, for an isolated system, the potential energy only depends upon r_{12}. For example,

$$U_{grav}(r_{12}) = \frac{-Gm_1m_2}{r_{12}} \text{ and the corresponding force is } \vec{F}_{grav1\leftarrow 2}(\vec{r}_{12}) = \frac{-Gm_1m_2}{r_{12}{}^2}\hat{r}_{12}$$

The textbook (at least the 1st printing) does <u>not</u> seem to give the gravitational constant, $G = 6.67\times 10^{-11}\ \mathrm{m^3/kg\cdot s^2}$. It is not needed for some calculations, but it is useful or necessary for others.

And the electric potential

$$U_{elect}(\vec{r}_1, \vec{r}_2) = \frac{1}{4\pi\varepsilon_o}\frac{q_1q_2}{r_{12}} \text{ and the corresponding force is } \vec{F}_{elect1\leftarrow 2}(\vec{r}_{12}) = \frac{1}{4\pi\varepsilon_o}\frac{q_1q_2}{r_{12}{}^2}\hat{r}_{12}$$

$$\frac{1}{4\pi\varepsilon_o} = 9\times 10^9\ Jm/C^2$$

We can describe a two-body system with either the Lagrangian approach or Newton's second law. We will use a little bit of each.

Of course, the Lagrangian is

$$\mathcal{L} = T - U = \tfrac{1}{2} m_1\dot{\vec{r}}_1^{\,2} + \tfrac{1}{2} m_2\dot{\vec{r}}_2^{\,2} - U(r_{12}).$$

$$\ldots$$

$$\mathcal{L}_{system} = \tfrac{1}{2} M\dot{\vec{R}}_{cm}^2 + \tfrac{1}{2}\mu\dot{\vec{r}}^2{}_{12} - U(r_{1,2})$$

Notice that there are only two (or 6) degrees of freedom: the position of the center of mass and the separation of the two objects from each other.

One way of conceptually subdividing this is

$$\mathcal{L} = T - U = \tfrac{1}{2} M\dot{\vec{R}}^2 + \left(\tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r)\right) = \mathcal{L}_{cm} + \mathcal{L}_{rel}.$$

There are no mixed terms, so there will be independent equations of motion for $\vec{R}$ and $\vec{r}$ just as if they described two non-interacting systems.

The Lagrangian for the CM is like that of a free particle (no force or potential) of mass *M*. In the first term,

$$\dot{\vec{R}}^2 = \dot{R}_x^2 + \dot{R}_y^2 + \dot{R}_z^2.$$

The coordinates R_x, R_y, and R_z are ignorable, so the equations for the center of mass motion are:

$$M\dot{R}_x = \text{constant} \quad M\dot{R}_y = \text{constant} \quad \text{and} \quad M\dot{R}_z = \text{constant},$$

or the total momentum of the system is constant:

$$\vec{P} = M\dot{\vec{R}} = \text{constant(vector)}.$$

The Lagrangian for the relative motion is like that of a particle of mass μ moving in a potential $U(r)$.

Note that $\frac{1}{2}\mu\dot{\vec{r}}^2$ isn't as simple as it looks – there are three coordinates:

In Cartesian: $\frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{1}{2}\mu\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$

In Polar/Cylindrical: $\frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{1}{2}\mu\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right)$

In Spherical: $\frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)$

<u>Equations of Motion</u>:

We already found that the equation of motion associated with the CM coordinate is:

$$\vec{P} = M\dot{\vec{R}} = \text{constant} \quad \text{or} \quad \dot{\vec{R}} = \text{constant}$$

When there are no external forces.

We are free to choose any inertial reference frame in which to analyze the relative motions of the two objects. In this case, a good choice is the CM frame where $\vec{R} = 0$ and $\dot{\vec{R}} = 0$, because the positions of the masses are described by:

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}_{12} \rightarrow \frac{m_2}{M}\vec{r}_{12},$$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}_{12} \rightarrow -\frac{m_1}{M}\vec{r}_{12}.$$

The total angular momentum of the system about its center of mass is:

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = m_1\vec{r}_1 \times \dot{\vec{r}}_1 + m_2\vec{r}_2 \times \dot{\vec{r}}_2,$$

$$\vec{L}_c = m_1\left(\frac{m_2}{M}\vec{r}_{12}\right)\times\left(\frac{m_2}{M}\dot{\vec{r}}_{12}\right) + m_2\left(\frac{-m_1}{M}\vec{r}_{12}\right)\times\left(\frac{-m_1}{M}\dot{\vec{r}}_{12}\right) = \frac{m_1m_2}{M^2}\left(m_2 + m_1\right)\left(\vec{r}_{12} \times \dot{\vec{r}}_{12}\right),$$

$$\vec{L}_c = \vec{r}_{12} \times \mu\dot{\vec{r}}_{12},$$

because $M = m_1 + m_2$ and $\mu = m_1m_2/M$. Of course, the rate of change of the angular momentum is $\frac{d\vec{L}_c}{dt} = \dot{\vec{r}}_{12} \times \mu\dot{\vec{r}}_{12} + \vec{r}_{12} \times \mu\ddot{\vec{r}}_{12} = 0 + \vec{r}_{12} \times \mu\ddot{\vec{r}}_{12}$

If we're dealing with a central force, then the last term is 0 since the two factors are parallel to each other.

So total angular momentum is conserved. Of course, if the angular momentum is constant than so is its *direction,* which is the direction of $\vec{r} \times \dot{\vec{r}}$, so $\vec{r}$ and $\dot{\vec{r}}$ remain in a fixed plane. If we call this the *xy* plane, $\vec{L}$ is along the *z* axis. We can use cylindrical polar coordinates, so:

$$\vec{r} = r\hat{r} \quad \text{and} \quad \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}.$$

Substituting these in gives:

$$\vec{L} = (r\hat{r}) \times \mu(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}) = (\mu r^2\dot{\phi})\hat{z},$$

With no external torques applied to the system, this angular momentum vector must be constant in both magnitude, $\ell = \mu r^2\dot{\phi}$, and direction.

This fact, in turn, means that r and v must be confined to the x-y plane (should one or the other venture out, then L would tip). So, we have essentially a 2-D problem.

From Monday, we have the Lagrangian for the two-body, central force problem:

$$\mathcal{L} = T - U = \tfrac{1}{2}M\dot{\vec{R}}^2 + \left(\tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r)\right) = \mathcal{L}_{cm} + \mathcal{L}_{rel}.$$

Note: the Lagrangian approach is powerful enough that if we *hadn't* first observed that angular momentum was conserved and that that meant all the action was confined to a plane, we could have started with

$$\tfrac{1}{2}\mu\dot{\vec{r}}^2 = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)$$

and found that to be the case. Here's how that argument goes.

Let's define our axes so that the initial position is on the z axis, so

θ_i=0.

Then imposing

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$

$$0 = \frac{d}{dt}\mu r^2\sin^2\theta\dot{\phi} \Rightarrow \mu r^2\sin^2\theta\dot{\phi} = const$$

But we know that initially it must be 0 since sin(0)=0, so it must always be 0. Unless the object *stays put* at θ_i=0, or r=0, this can only be if $\dot{\phi} = 0$ always.

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$\mu r\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - \frac{\partial U}{\partial r} = \mu\ddot{r}$$

but $\dot{\phi} = 0$, so

$$\mu r\dot{\theta}^2 - \frac{\partial U}{\partial r} = \mu\ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}}$$

$$\mu r^2 \sin\theta\cos\theta\dot{\phi}^2 = \frac{d}{dt}\mu r^2\dot{\theta}$$

but $\dot{\phi} = 0$, so $0 = \dfrac{d}{dt}\mu r^2\dot{\theta}$ so

$$l = \mu r^2\dot{\theta}$$

In the CM frame where $\dot{\vec{R}} = 0$, the CM part of the Lagrangian is zero:

$$\mathcal{L} = \mathcal{L}_{rel} = \tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r),$$

or in terms of cylindrical polar coordinates (since this is a 2-D problem):

$$\mathcal{L} = \mathcal{L}_{rel} = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2 + \cancel{\dot{z}^2}\right) - U(r).$$

The Lagrange equation associated with ϕ is:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}},$$

but the left hand side is zero so the ϕ equation is:

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2\dot{\phi} = \text{constant} = \ell .$$

(which we'd already deduce)

The Lagrange equation associated with r is (the *radial equation*):

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}},$$

$$\boxed{\mu r\dot{\phi}^2 - \frac{\partial U}{\partial r} = \mu\ddot{r}}.$$

Example from section 8.3 problems

Like Problem 8.9: two masses, m_1 and m_2, joined by a spring of relaxed length r_o and stiffness k lying in an x-y plane; do it in polar, c.m. coordinates. A) find the equations of motion for r and ϕ, and express them in terms of the constant angular momentum, *l*.

$$\mathcal{L} = \mathcal{L}_{rel} = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \tfrac{1}{2}k\left(r - r_o\right)^2$$

We can adopt everything from above but with this particular potential, $U = \tfrac{1}{2}k\left(r - r_o\right)^2$

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}}$$

$$\boxed{\begin{aligned} &\mu r\dot{\phi}^2 - \frac{\partial U}{\partial r} = \mu\ddot{r} \\ &\mu r\dot{\phi}^2 - k\left(r - r_o\right) = \mu\ddot{r} \end{aligned}}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \mu r^2\dot{\phi} = \text{constant} = \ell$$

B) What is the equilibrium spring length in terms of the rotational rate?

At equilibrium,

$$\mu r\dot{\phi}^2 - k\left(r - r_o\right) = \mu\ddot{r} = 0 \Rightarrow r\left(\mu\dot{\phi}^2 - k\right) + kr_o = 0 \Rightarrow r_{eq} = \frac{kr_o}{\left(k - \mu\dot{\phi}^2\right)} = \frac{r_o}{\left(1 - \dfrac{\mu\dot{\phi}^2}{k}\right)}$$

Not surprisingly, the spinning system has a longer equilibrium length. You'll also observe that the equilibrium separation blows up as $\dot{\phi} \to \sqrt{\dfrac{k}{\mu}}$

Equivalent 1-D Problem:

Solving the ϕ equation for $\dot{\phi}$ gives:

$$\dot{\phi} = \frac{\ell}{\mu r^2},$$

so the radial equation can be written purely in terms of *r* and constants (like the angular momentum):

$$\mu\ddot{r} = -\frac{\partial U}{\partial r} + \mu r\left(\frac{\ell}{\mu r^2}\right)^2,$$

$$\mu\ddot{r} = -\frac{\partial U}{\partial r} + \frac{\ell^2}{\mu r^3}.$$

The first term on the right hand side is the actual force $F = -\partial U/\partial r$, which points in the radial direction. The second term on the right, $F_{\mathrm{cf}} = \dfrac{\ell^2}{\mu r^3} = \mu r\dot{\phi}^2$, is a "fictitious force" that points outward called the *centrifugal force*. This isn't a real force, as in, there isn't really something pulling or pushing on the system; but, the role it plays in determining just how the radial component of the motion changes is the same as a real force could do. (as with the example we just did – the faster the system spins, the more the parts are flung apart – that's what this term is about.)

Looking at the energy expression,

$$E = T + U = \tfrac{1}{2}\mu v^2 + U(r) = \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu(r\dot{\phi})^2 + U(r) = \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) = \tfrac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

The kinetic energy associated with rotation can be rewritten to show that it depends only on separation, just like a potential energy does, this term could be referred to as a "fictitious potential" called a *centrifugal potential.*

$$U_{cf}(r) = \frac{\ell^2}{2\mu r^2}$$

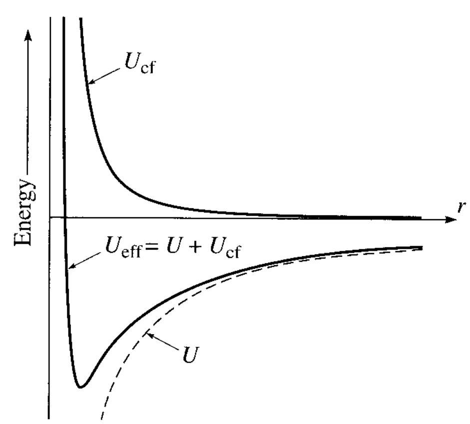
$$E = \tfrac{1}{2}\mu\dot{r}^2 + U_{cf}(r) + U(r) = \tfrac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

Of course, if you take its derivative, you get back the centrifugal force.

$$F_{cf} = -\frac{\partial}{\partial r}U_{cf}$$

$$\frac{\ell^2}{\mu r^3} = -\frac{\partial}{\partial r}\left(\frac{\ell^2}{2\mu r^2}\right)$$

The radial motion is exactly the same as that of a particle of mass μ moving in a one dimensional potential $U_{\mathrm{eff}} = U + U_{\mathrm{cf}}$. Graphs of the potential energies (assuming an attractive 1/r real potential) are shown below.

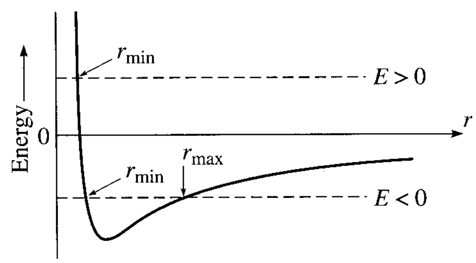


Therefore, we can use the expression:

$$\tfrac{1}{2}\mu\dot{r}^2 + U_{\text{eff}} = E,$$

to learn about the objects radial motion ($\tfrac{1}{2}\mu\dot{r}^2$ involves only one component of the velocity, but KE does <u>not</u> have components!). The radial motion is described *as if* the particle was moving in just one dimension with the potential U_{eff}. This same effective potential comes up when the real potential is gravitational or electrical. Of course, if we're talking electrons and protons interacting, we really should change the energy relation into a wave equation for the wavefunction, but the same basic terms, including the effective potential, appear.

We can learn something about the motion from an energy diagram like the one shown below.



- If $E < 0$ and $E > (U_{\text{eff}})_{\text{min}}$, there are two turning points, r_{min} and r_{max}, so the motion is *bounded.*
- If the energy is equal to the minimum value of U_{eff}, then $\dot{r} = 0$ and the radius is constant. That means the orbit is circular.
- If $E > 0$, there is just one turning point, r_{min}, so the object can move off to infinity. The orbit is *unbounded* in this case.

<u>Example</u>: Potential for Nuclear Fusion

Example from 8.4 problems

8.13: Now look at the two masses on a spring's effective potential, sketch it. Equilibrium separation. Frequency. (see paper notes)

8.14: What about the more general case, U=kr^n, what's the equilibrium separation, and is it stable (will find that it's stable only for n > -2; see paper notes)

Next two lectures, we'll discuss the possible shapes of an "orbit" (that name is used even if the motion is unbounded).