Physics 331: Euler's Equations

Fri., 12/7	**10.7-.8** Euler's Equations	
Mon., 12/10	**Review for Final**	**Project**

Handouts:

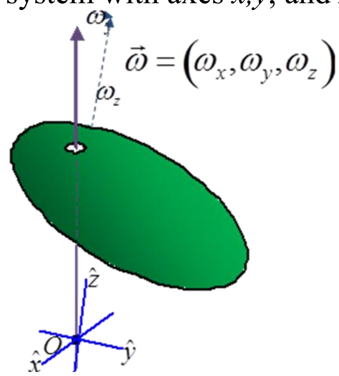
- Exam #3 Equation Sheet and sample test
- Rubber bands for demo

10.7 Euler's Equations:

Now we want to start talking about *dynamics* – how interactions change the rotational motion of an object. Of course, in terms of an *inertial* reference frame, "Newton's 2nd Law" of rotation says how the torque causes angular momentum to change.

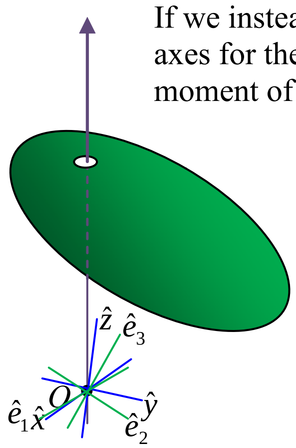
$$\left(\frac{d\vec{L}}{dt}\right)_{S_o} = \dot{\vec{L}}_o = \vec{\Gamma}_{net}$$

While we can describe an object's rotation about some point in terms of any old coordinate system with axes *x,y*, and *z*, that generally means dealing with a complicated moment of inertial:



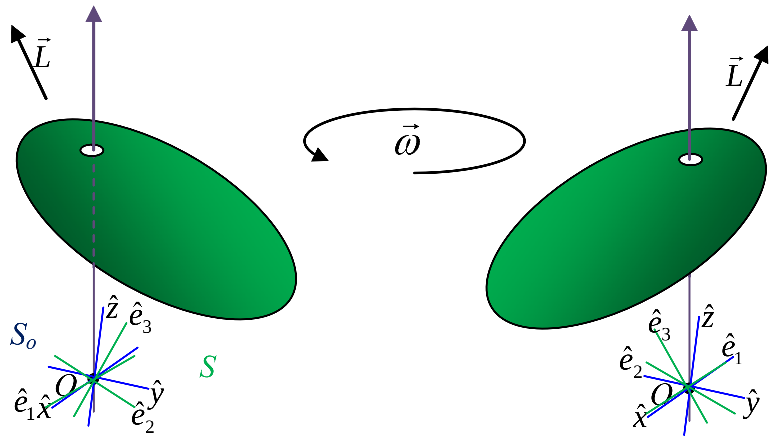
$$\overleftrightarrow{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

If we instead describe that motion in terms of a coordinate system that's defined by the *principle* axes for the object about that point (note: it is point dependent), we can use a much simpler moment of inertia, and often get more conceptual insight.



$$\overleftrightarrow{I} = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$$

Unfortunately, the *nice* coordinate system in which to work, that defined by the principle axes, rotates with the rotating object, and so *doesn't* describe an *inertial* reference frame – it describes a *rotating* reference frame.



Fortunately, in the previous chapter we dealt with something quite similar – relating how we describe the time derivative of the same vector in terms of its projection on the inertial frame's axes and on the rotating frame's axes. We saw that in general, for any old vector, Q,

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_o} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\omega}_{frame} \times \vec{Q}.$$

If you recall, we'd used this relationship for relating the position vector's time derivative, velocity, in the two frames and relating the velocity's time derivative, acceleration, in the two frames. Now we want to relate angular momentum's time derivative:

$$\left(\frac{d\vec{L}}{dt}\right)_{S_o} = \left(\frac{d\vec{L}}{dt}\right)_S + \vec{\omega} \times \vec{L},$$

$$\dot{\vec{L}}_o = \dot{\vec{L}} + \vec{\omega} \times \vec{L}$$

where the derivative of the angular momentum in the (noninertial) rotating body frame (fixed in the body) is $\dot{\vec{L}}$. Combining these two expressions yields *Euler's equation* (expressed in the body frame):

$$\vec{\Gamma} = \dot{\vec{L}} + \vec{\omega} \times \vec{L}.$$

To parallel our work in Chapter 9, we could write this as

$$\vec{\Gamma} + \vec{\Gamma}_{frame} = \dot{\vec{L}}$$

And call $\vec{\Gamma}_{frame} \equiv \vec{L} \times \vec{\omega}$ a "frame torque" – there's no real interaction here, but the angular momentum determined in this frame changes *as if* subject to such a torque.

Note: because ω itself can change with time, this is a bit trickier to conceptualize than in the simple case of a constant ω that we considered in Ch. 9.

Okay, so in terms of the principal axes, the angular momentum in the rotating body frame is simply:

$$\vec{L} = (I_{11}\omega_1, I_{22}\omega_2, I_{33}\omega_3).$$

And so the cross product is

$$\vec{\omega} \times \vec{L} = \det \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_{11}\omega_1 & I_{22}\omega_2 & I_{33}\omega_3 \end{vmatrix} = \hat{e}_1 (\omega_2 I_{33}\omega_3 - \omega_3 I_{22}\omega_2) + \hat{e}_2 (\omega_1 I_{33}\omega_3 - \omega_3 I_{11}\omega_1) + \hat{e}_2 (\omega_1 I_{22}\omega_2 - \omega_1 I_{11}\omega_2)$$

$$\vec{\omega} \times \vec{L} = \hat{e}_1 (I_{33} - I_{22})\omega_2\omega_3 + \hat{e}_2 (I_{33} - I_{11})\omega_1\omega_3 + \hat{e}_2 (I_{22} - I_{11})\omega_1\omega_2$$

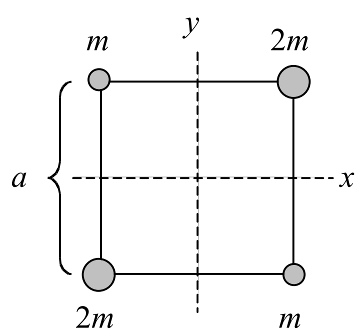
So, written in component form *Euler's equations* are:

$$\hat{e}_1: \quad I_{11}\dot{\omega}_1 - (I_{22} - I_{33})\omega_2\omega_3 = \Gamma_1,$$

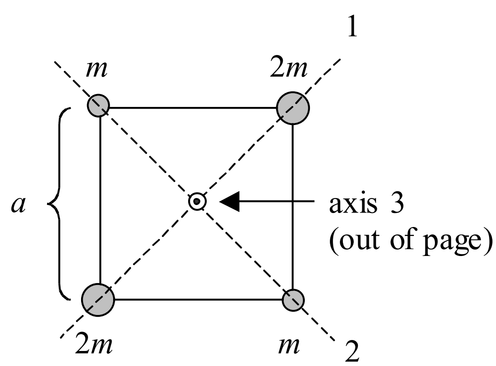
$$\hat{e}_2: \quad I_{22}\dot{\omega}_2 - (I_{33} - I_{11})\omega_3\omega_1 = \Gamma_2,$$

$$\hat{e}_3: \quad I_{33}\dot{\omega}_3 - (I_{11} - I_{22})\omega_1\omega_2 = \Gamma_3.$$

Example Torque but constant angular velocity: Suppose the object below with a constant angular velocity about the x axis, $\vec{\omega} = \omega\hat{x}$. What is the torque on the object?



From before, the principal axes are the ones shown below.



The principal moments are:

$$I_1 = ma^2, \quad I_2 = 2ma^2, \quad \text{and} \quad I_3 = 3ma^2.$$

Along

$$\hat{e}_1 = \tfrac{1}{\sqrt{2}}(1,1,0),\ \hat{e}_2 = \tfrac{1}{\sqrt{2}}(1,-1,0),\ \hat{e}_3 = (0,0,1)$$

Or flipping that around

$$\hat{x} = \tfrac{1}{\sqrt{2}}(\hat{e}_1 + \hat{e}_2)$$

So,

The angular velocity is $\vec{\omega} = \omega\hat{x} = (\omega/\sqrt{2})\,\hat{e}_1 + (\omega/\sqrt{2})\,\hat{e}_2$

or $\omega_1 = \omega/\sqrt{2}$, $\omega_2 = \omega/\sqrt{2}$, and $\omega_3 = 0$.

In our scenario, we're applying whatever torque is necessary to keep this rotation constant, so these do <u>not</u> change with rotation, so:

$$\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0.$$

The time derivatives of all of the components are zero. Euler's equations are:

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = \Gamma_1,$$

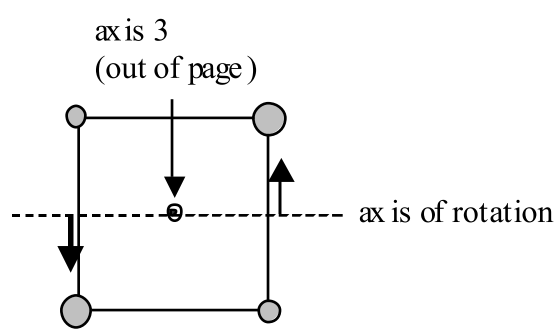
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = \Gamma_2,$$

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = \Gamma_3,$$

so the components of the torque are $\Gamma_1 = \Gamma_2 = 0$ (because $\omega_3 = 0$ and $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$) and:

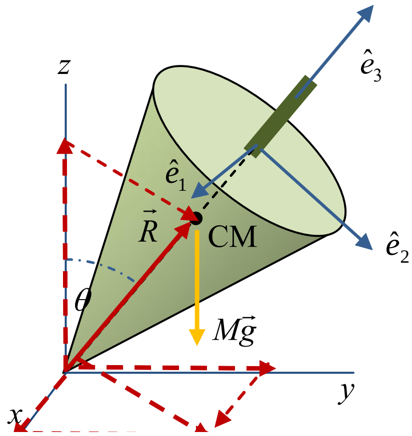
$$\Gamma_3 = -(I_1 - I_2)\omega_1\omega_2 = (I_2 - I_1)\omega_1\omega_2 = \frac{ma^2\omega^2}{2}.$$

This torque is the result of forces on the bearings in the directions shown in the diagram below.



Qualitatively, while we're forcing it to spin about the x axis, it would naturally spin about axis e_2, that's the one that would have the most massive objects moving the farthest (this is analogous to the way a spinning skater's arms get flung out unless she forces them to stay in – maximizing the moment of inertia). So the torque that is applied is what's required to oppose the counter-clockwise 45° rotation about the z / e_3 axis which would then have the e_2 axis aligned with the x axis.

Example: For a top that is symmetric about the axis $\hat{e}_3$, the other two moments are equal: $I_{11} = I_{22}$. Suppose the top is spinning about $\hat{e}_3$ and tilted as shown below.



The torque is always perpendicular to $\vec{R}$ and thus $\hat{e}_3$, so $\Gamma_3 = 0$. This means that:

$$I_{33}\dot{\omega}_3 = 0,$$

so the angular velocity about the symmetry axis doe <u>not</u> change ($\omega_3 = \text{constant}$). We used this yesterday.

Zero Torque

A <u>special case</u> is Euler's equations with zero torque, which are:

$$I_{11}\dot{\omega}_1 = (I_{22} - I_{33})\omega_2\omega_3,$$

$$I_{22}\dot{\omega}_2 = (I_{33} - I_{11})\omega_3\omega_1,$$

$$I_{33}\dot{\omega}_3 = (I_{11} - I_{22})\omega_1\omega_2.$$

Let's look at two cases of *free precession* (no torque).

Two Equal Principal Moments:

Suppose $I_{11} = I_{22}$. The last Euler equation with zero torque simplifies to:

$$I_{33}\dot{\omega}_3 = (I_{11} - I_{22})\omega_1\omega_2 = 0,$$

so:

$$\dot{\omega}_3 = 0 \quad \text{and} \quad \omega_3 = \text{constant}.$$

Define the constant frequency:

$$\Omega_b \equiv \frac{(I_{11} - I_{33})\omega_3}{I_{11}},$$

then the other two of Euler's equations with zero torque can be written as:

$$\dot{\omega}_1 = \frac{(I_{22} - I_{33})\omega_3}{I_{11}}\omega_2 = \Omega_b\omega_2,$$

$$\dot{\omega}_2 = \frac{(I_{33} - I_{11})\omega_3}{I_{22}}\omega_1 = -\Omega_b\omega_1.$$

Differentiate the first and substitute in the second to get:

$$\ddot{\omega}_1 = -\Omega_b^2\omega_1.$$

If we choose $\omega_1 = \omega_o$ and $\omega_2 = 0$ at $t = 0$, then the solution is:

$$\omega_1 = \omega_o \cos\Omega_b t.$$

Take the derivative of this solution and substitute that into the equation for ω_2 above to get:

$$\omega_2 = -\omega_o \sin\Omega_b t,$$

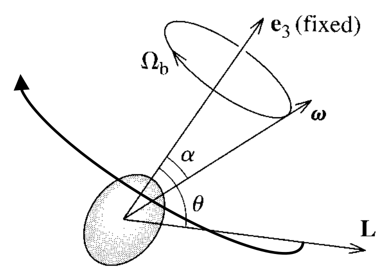
so:

$$\vec{\omega} = (\omega_o \cos\Omega_b t, -\omega_o \sin\Omega_b t, \omega_3).$$

The angular momentum is:

$$\vec{L} = (I_{11}\omega_o \cos\Omega_b t, -I_{11}\omega_o \sin\Omega_b t, I_{33}\omega_3).$$

So as measured against e_1, e_2, and e_3, the angular velocity and angular momentum rotate or precess about the e_3 axis with frequency Ω_b.



(a) Body frame

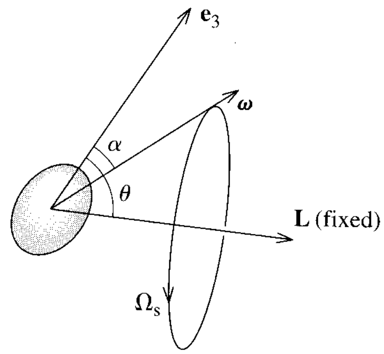
Inertial Frame Perspective. But what does it look like in the inertial frame, say, from the perspective of someone who tosses the object in the air and watches it spin and wobble?

The hook is that, in the absence of an external torque, we know that

$$\dot{\vec{L}}_o = 0$$

So measured against the inertial x,y,z, the angular momentum must be seen to be constant.

In the space frame, $\hat{e}_3$ and $\vec{\omega}$ rotate around $\vec{L}$ (fixed because it's conserved) with an angular frequency $\Omega_s = L/I_{11}$ (Problem 10.46).



(b) Space frame

The *Chandler wobble* (period of about 400 days) of the earth is an example of this type of free precession that occurs because the earth is spheroidal (bulges slightly at the equator), not spherical.

Three Different Principal Moments: ("Tennis Racket Theorem")

(Barger & Olsson approach, p. 253) Assume $I_{11} < I_{22} < I_{33}$

Define the <u>positive</u> constants:

$$\eta_1 = \frac{I_{33} - I_{22}}{I_{11}},$$

$$\eta_2 = \frac{I_{33} - I_{11}}{I_{22}},$$

$$\eta_3 = \frac{I_{22} - I_{11}}{I_{33}},$$

and rewrite Euler's equations without torque as:

$$\dot{\omega}_1 = -\eta_1 \omega_2 \omega_3,$$

$$\dot{\omega}_2 = +\eta_2 \omega_3 \omega_1,$$

$$\dot{\omega}_3 = -\eta_3 \omega_1 \omega_2.$$

Suppose the object is rotating mostly about the axis of the intermediate moment of inertia:

$$\vec{\omega} \approx \omega_2 \hat{e}_2, \quad \omega_1, \omega_3 << \omega_2,$$

then $\omega_2 \approx \text{constant}$ because $\omega_1 \omega_3$ is very small (product of two small numbers). Treat ω_2 as a constant in the first and third equations:

$$\dot{\omega}_1 = -(\eta_1 \omega_2)\omega_3,$$

$$\dot{\omega}_3 = -(\eta_3\omega_2)\omega_1 .$$

This is similar to what we had for two equal moments, except the signs are both negative. Differentiate the first equation and substitute in the second to get:

$$\ddot{\omega}_1 = +(\eta_1\eta_3\omega_2^2)\omega_1 ,$$

which has the solution:

$$\omega_1(t) = Ae^{\omega_2\sqrt{\eta_1\eta_3}\,t} + Be^{-\omega_2\sqrt{\eta_1\eta_3}\,t} .$$

Plug this into the equation relating $\dot{\omega}_1$ and ω_3 to get:

$$\omega_3(t) = \frac{\omega_2\sqrt{\eta_1\eta_3}}{\eta_1\omega_2}\left[Ae^{\omega_2\sqrt{\eta_1\eta_3}\,t} + Be^{-\omega_2\sqrt{\eta_1\eta_3}\,t}\right] = \sqrt{\frac{\eta_3}{\eta_1}}\left[Ae^{\omega_2\sqrt{\eta_1\eta_3}\,t} + Be^{-\omega_2\sqrt{\eta_1\eta_3}\,t}\right].$$

The rotation about the axis with the intermediate moment is <u>unstable</u> because rotations about the other two axes grow exponentially. The approximation that $\omega_1, \omega_3 << \omega_2$ is <u>not</u> valid for long.

Suppose the object is rotating mostly about the axis of the smallest moment of inertia:

$$\vec{\omega} \approx \omega_1\hat{e}_1, \quad \omega_2, \omega_3 << \omega_1 ,$$

then $\omega_1 \approx \text{constant}$ because $\omega_2\omega_3$ is very small (product of two small numbers). Treat ω_1 as a constant in the first and third equations:

$$\dot{\omega}_2 = +(\eta_2\omega_1)\omega_3 ,$$

$$\dot{\omega}_3 = -(\eta_3\omega_1)\omega_2 .$$

These combine to give:

$$\ddot{\omega}_2 = -(\eta_2\eta_3\omega_1^2)\omega_2 ,$$

which has the solution:

$$\omega_2(t) = C\sin\left(\omega_1\sqrt{\eta_2\eta_3}\,t - \delta\right).$$

The other solution is:

$$\omega_3(t) = \sqrt{\frac{\eta_3}{\eta_2}}\,C\cos\left(\omega_1\sqrt{\eta_2\eta_3}\,t - \delta\right).$$

The rotation about the axis with the smallest moment is <u>stable</u> because rotations about the other two axes oscillate. The same is also true of rotation about the axis with the largest moment.

<u>DEMO</u>: The results above are often called the *Tennis Racket Theorem*. The rotation about the principal axes with the smallest and largest moments of inertia are stable and rotation about the principal axis with the intermediate moment of inertia is unstable. This can be <u>demonstrated</u> with a book by putting a rubber band around it to keep it closed.