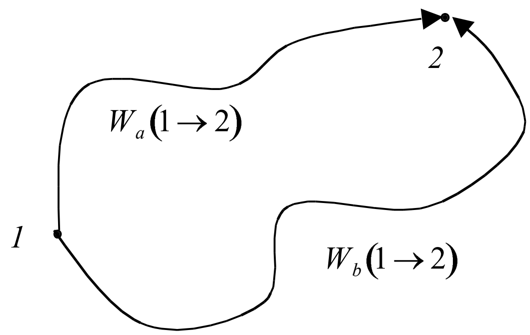
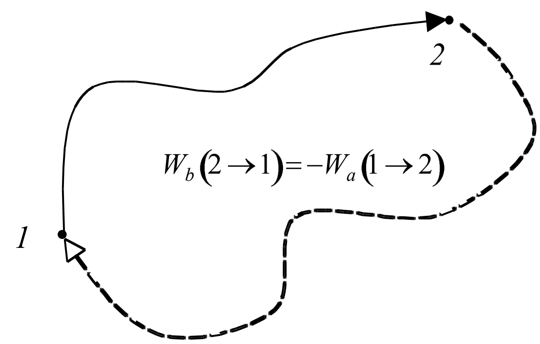
Thurs. 9/30 Fri., 10/1	**4.4-.6** Curl of Conservative Force, Varying Potential, 1-D systems	**HW3,** **Project Topic**
Mon. 10/4 Wed. 10/6 Thurs. 10/7 Fri. 10/8	**4.7-.8** Curvilinear 1-D, Central Force **4.9** Energy of 2 Particle Interaction **5.1-.3 (2.6)** Hooke's Law, Simple Harmonic (Complex Sol'ns)	**HW4**

Announcement: SPS Comet Observing trip with CSUSB

Path-Independence of Work:

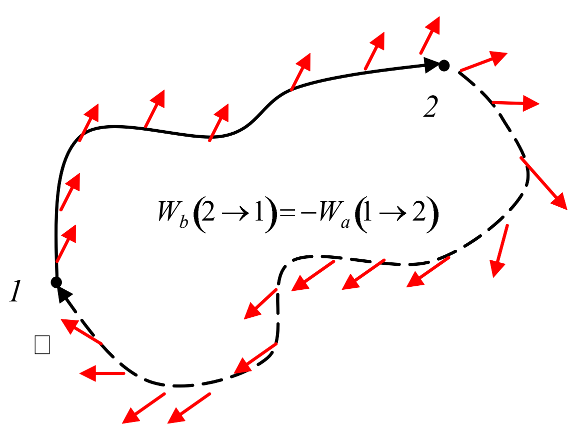
The condition that the line integral of the force is path independent is equivalent to the condition that the line integral for any closed loop is zero as is illustrated below. In the diagram below, if $W_a(1\rightarrow 2) = W_b(1\rightarrow 2)$, then $W(\text{closed loop}) = 0$ because $W_b(2\rightarrow 1) = -W_b(1\rightarrow 2)$.





$$\text{Then } W_{1\rightarrow 1} = \int_{\text{closed loop}} \vec{F}\cdot d\vec{r} = 0$$

Think about what it would take for this to *not* be the case. Say, on the upper branch the force points up and a little to the *right,* and on the lower branch it points down and a little to the *left.* Then positive work is done headed along the upper branch from 1 to 2 *and* positive work is done headed along the lower branch from 2 to 1 clearly those two positive works aren't going to add to 0. Visualize that force as a function of position, it might look something like



That force 'curls' around from pointing one way to pointing another – clearly the F_x depends on the y component of the position.

Clearly, if a force curls around like this, then the work is path dependent, and the work done in traveling a closed path isn't necessarily 0.

Now we'll make that idea mathematically concrete. Crossing the del operator into a force would mean 'multiplying' say d/dy by F_x, i.e., seeing how much the F_x component of the force depends on the y-component of the location; looking at the illustration above, it's that kind of dependence that characterizes 'curl.'

So, we define

$$curl.of.f = \vec{\nabla} \times \vec{F}$$

$$\vec{\nabla} \times \vec{F} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_y \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \left(\partial F_z/\partial y - \partial F_y/\partial z\right)\hat{x} + \left(\partial F_x/\partial z - \partial F_z/\partial x\right)\hat{y} + \left(\partial F_y/\partial x - \partial F_x/\partial y\right)\hat{z}$$

(if one must, and sometimes, indeed one must, one can translate this into polar or spherical coordinates; see the back inside cover)

As you might then expect, if this doesn't evaluate to 0, if the force *has* curl, then the work *is* path dependent.

What you might not guess is that this operation directly relates to the closed-path work integral in a really *simple* way: from vector calculus, Stoke's Theorem is

$$\int_{\text{closed loop}} \vec{F} \cdot d\vec{r} = \int_{\text{area enclosed}} \left(\vec{\nabla} \times \vec{F}\right) \cdot \hat{n}\, dA,$$

where $\hat{n}$ is normal to the area. Those of you who've already had E&M have seen this proven, those of you who haven't yet, have something to look forward to. It's a fun derivation, but we've got other fish to fry today.

Taking it as a given, then if and only if the *curl* of a force vanishes:

$$\vec{\nabla} \times \vec{F} = 0,$$

is the work path-independent. If the force *also* depends on only the position, it is conservative and a potential energy can be defined.

So, let's play with this.

Exercise: Which of the following is/are a conservative force?

$$\vec{F}_a = y\hat{x} - x\hat{y} + z^2\hat{z}$$

$$\vec{F}_a = y\hat{x} + x\hat{y} + z^3\hat{z}$$

The curls of the forces are:

$$\vec{\nabla}\times\vec{F}_a=\det\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y & -x & z^2\end{bmatrix}=[(-1)-(1)]\hat{z}=-2\hat{z}$$

$$\vec{\nabla}\times\vec{F}_b=\det\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y & x & z^3\end{bmatrix}=[(1)-(1)]\hat{z}=0$$

The second force is conservative and the first one is not.

4.5 Time Dependent "Potential Energy"

The only point I want to make here is that the author unfolds this story in an unnecessarily mysterious way and part of that has to do with his poor choice of associating potential energy with a single object rather than the two interacting objects and his completely overlooking the step of defining his system. The potential energy is shared by the charges on the conducing sphere and the distant charge and also within that system are all the interactions of the charges on the sphere with each other. Looking at that system, the slow reconfiguring of the charges that were initially on the sphere changes the potential energy terms for their interactions with the distant charge, but if nothing external is interacting with the system, then the total energy in the system is constant.

Alternatively, if you look at the distant charge as the system, then its interaction with the charges on the sphere is an *external* one, and thus not validly represented as a "potential energy" of the charge. The external object does *work* on the charge, and its kinetic energy (the only kind it classically can have all to itself) changes.

4.6 Linear 1-D Systems:

Of course, everything's simpler in 1-D. If a particle is constrained to move along a line (call it the x axis), work and energy are simple. The work done by a force is an ordinary integral:

$$W(x_1\to x_2)=\int_{x_1}^{x_2}F_x(x)\,dx.$$

As long as a force <u>only</u> depends on the position x (and not velocity v or time t), it is conservative! If we choose $U(x_o)=0$ at the reference point x_o, then the associated potential energy is:

$$U(x)=-\int_{x_o}^{x}F_x(x')\,dx'.$$

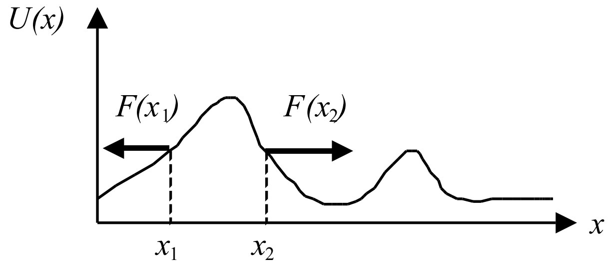
Relative to that at the reference point. (Note: like moment of inertia, angular momentum, and position, potential energy values are *relative.*)

(assuming that the other party to the interaction implied by that force isn't moving too)

The force in 1-D is related to the potential by:

$$F_x = -\frac{dU}{dx}.$$

Graphically, you can think of a plot of the potential energy vs. position as a "roller coaster track." There will always be a force on an object in the downhill direction (see diagram below).



A particle is in *equilibrium* if the net force on it is zero. $F_x = 0$ when $dU/dx = 0$, which corresponds to a minimum or maximum of $U(x)$ vs. x. An equilibrium is *stable* if a small displacement from it results in a force back toward the equilibrium. This occurs where $U(x)$ is a minimum and $d^2U/dx^2 > 0$. An equilibrium is *unstable* if a small displacement from it results in a force away from the equilibrium. This occurs where $U(x)$ is a maximum and $d^2U/dx^2 < 0$. (A saddle point where $dU/dx = 0$ and $d^2U/dx^2 = 0$ is unstable because a small displacement in one direction will result in a force that pushes the particle further away.)

For a system interacting by only conservative forces, the total energy:

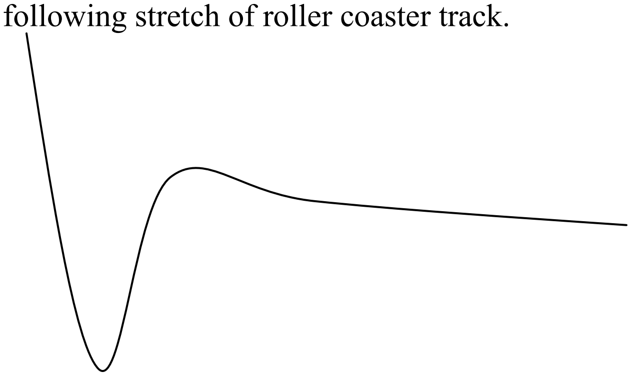
$$E = T + U(x)$$

is conserved.

Points where $U(x) = E$ are known as *turning points*. At these points, $T = 0$ and $v = 0$. Interacting particle will not separate beyond this distance, to locations where U is larger, because conservation of energy would require that the kinetic energy be *negative*. Regions where $U(x) \le E$ are classically allowed and those where $U(x) > E$ are classically forbidden.

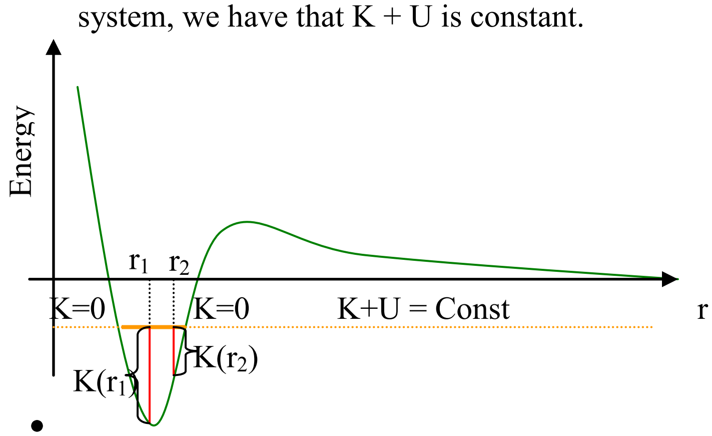
4.4.1 Energy diagrams

- For the time being, this relation is most useful in that it helps us to interpret energy diagrams.
- **Rollercoaster Track.**
  - An energy diagram is like a roller coaster track. By just looking at a rollercoaster track, you can tell how the cars will move – where they'll speed up and where they'll slow down, and where, if it weren't for a chain pulling them, they would stop and roll backwards. It's the same with a energy diagram. Say you saw the following stretch of roller coaster track.



- **Q:** If you imagine placing the cart high on the left slope, what would it do, what direction would it go, where would it be speeding up, where would it be slowing down?
    - **A:** Speed-up to the right until it hits the bottom, then slow down on it's way further right until it hits the little peak, then gradually speed up as it goes right.
- **Q:** What is the lowest starting point on the left that would allow the cart to still clear the hump on the right?
    - **A:** Just even with the hump.
- **Potential Energy Curve.** If, instead, this curve plotted the gravitational potential energy of the rollercoaster (which varies proportional to elevation after all), then what would the Force – Potential Energy relation say?
    - On the far right, where the slope of the potential energy is strong and negative, what should the force be like?
        - The force is large and positive (to the right).
    - Where the slope is 0?
        - The force is 0: an equilibrium point.
    - Where the slope is positive?
        - The force is negative (to the left).
    - Moral: Your intuition about this rollercoaster track holds perfectly for a plot of potential energy vs. position.

- **Nuclear Potential.** The potential energy of a nucleus and a proton plots like this. If the proton stays near in, it is attracted by the strong force, but if it gets far enough out, the strong force tapers off, but it still feels the electric repulsion of all that other protons.
- Since $K + U = -mc^2 + E$ and mc^2 is generally constant and E is constant in an isolated system, we have that K + U is constant.



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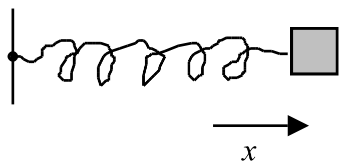
04-potential-energy-well.py Choose different K+U lines, see how the kinetic energy varies as the particle moves across the potential contour.

- **Bound States =** we say a particle is in a bound state if it can't "escape the potential well." This is the case for the protons and neutrons in nuclei. One would require additional energy to surmount the barrier and escape. Similarly, any ball you can throw is gravitationally bound to the Earth – it will only get a few meters into the air before its kinetic energy comes to 0, and it falls back down again.

- In this way, the idea of energy helps us to see what a system can and cannot do.

<u>Example 1</u>: (Prob. 4.9) The force exerted by an ideal spring with its left end fixed is $F(x) = -kx$, where the spring is unstretched at $x = 0$. The parameter k is called the *spring constant*. (a) If we choose $U = 0$ at the equilibrium, what is the corresponding potential energy? (b) Suppose a spring is hung vertically from the ceiling with a mass m attached to the other end and constrained to move vertically. If y is the displacement downward from the equilibrium position, find the total potential energy.

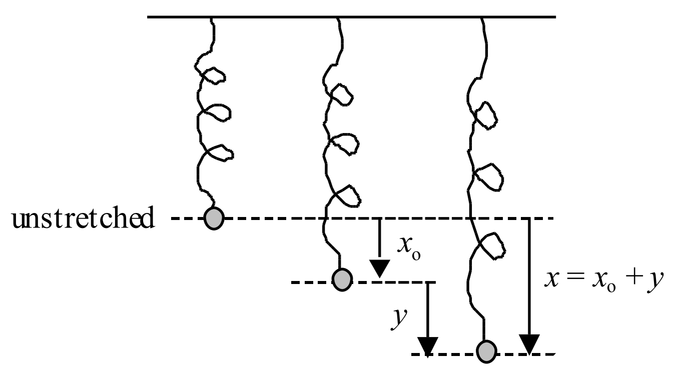
(a) If $x > 0$ ($x < 0$), the spring is stretched (compressed) exerts a force to the left (right).



The potential energy is:

$$U_{spr}(x) = -\int_{x_o}^{x} F_x(x')\,dx' = \int_{0}^{x} kx'\,dx' = \tfrac{1}{2}kx^2$$

(b) Let x_o be the equilibrium position and y be the distance from equilibrium as shown below.



In equilibrium, the spring force and weight must balance so $kx_o = mg$ or $x_o = mg/k$. The total potential energy is (for gravity, it decreases as the mass moves downward):

$$U = \tfrac{1}{2}kx^2 - mgx = \tfrac{1}{2}k(x_o + y)^2 - mg(x_o + y)$$

$$U = \tfrac{1}{2}ky^2 + (kx_o - mg)y + \left(\tfrac{1}{2}kx_o^2 - mgx_o\right)$$

$$U = \tfrac{1}{2}ky^2 + \text{constant}$$

The constant has no physical consequence, so the behavior is the same as for a mass attached to a spring that moves horizontally without friction. You could define away the constant.

Example 2: A 2-kg particle moves in one dimension under a force:

$$F(x) = -bx + 2c\sin(ax),$$

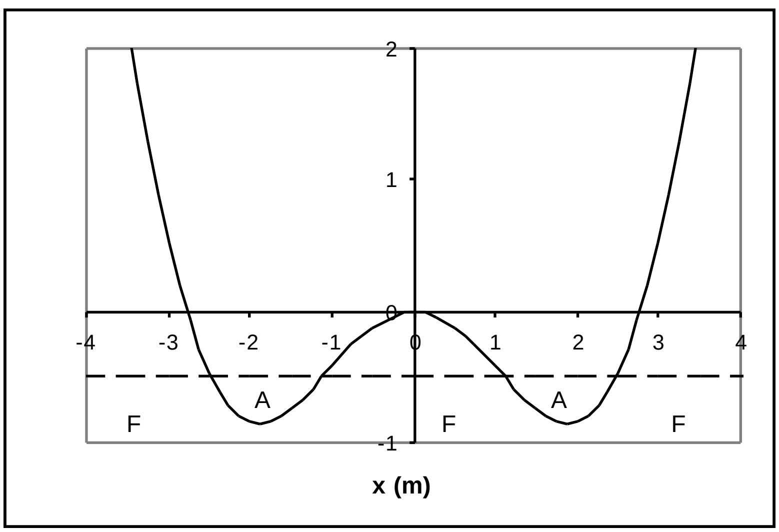
where a = 1 m^{-1}, b = 1 N/m, and c = 1 N. The argument of the sine is in radians. (a) Find the potential energy with the reference point at the origin so that $U(0) = 0$. Sketch the potential and show the classically allowed and forbidden regions if the total energy is E = -0.5 J. (b) Identify the three points of equilibrium and determine if each is stable or unstable.

(a) The potential is found by integrating the force (with a minus sign!):

$$U(x) = -\int_0^x F(x')\,dx' = -\int_0^x \left[-bx' + 2c\sin(ax')\right] dx',$$

$$U(x) = \left[\frac{bx'^2}{2} + \frac{2c}{a}\cos(ax')\right]_0^x = \left(\frac{bx^2}{2} + \frac{2c}{a}\cos(ax) - \frac{2c}{a}\right).$$

The constants a, b, and c are all one and each term is in Joules when x is in meters. The graph below shows $U(x)$ vs. x. For large x, the x^2 dominates the oscillating term. The dashed line is E = -0.5 J and the allowed (A) and forbidden (F) regions are labeled. The particle is only allowed to be where $E > U(x)$ because $E = T + U(x)$ and T must be positive.



(b) The points of equilibrium are where $F(x) = -dU/dx = 0$. This gives the transcendental equation:

$$x = \frac{2c}{b}\sin(ax)$$

The solution $x = 0$ corresponds to an unstable equilibrium because $d^2U/dx^2 < 0$. The other two solutions can be found approximately by making successive guesses to get $x \approx \pm 1.896$, which are stable because $d^2U/dx^2 > 0$.

If energy is conserved, $E = T + U(x)$, then:

$$T = \tfrac{1}{2} m\dot{x}^2 = E - U(x),$$

which can be used to find the velocity as a function of position:

$$\dot{x}(x) = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}.$$

The velocity is $\dot{x} = dx/dt$, so $dt = dx/\dot{x}$. This can be integrated to find the time for motion between two points:

$$t = \int_{x_0}^{x} \frac{dx'}{\dot{x}(x')} = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}}.$$

In practice, this can be difficult to calculate because the integrand goes to infinity as it approaches the turning point where $\dot{x} = 0$. Even for the simple pendulum, there is no analytical solution (see Prob. 4.38). Energy conservation is typically <u>not</u> a good way to get information about time.

<u>Example 3</u>: (2.10 of Fowles & Cassiday 5th ed.) A particle of mass m is released from rest at $x = b$ and its potential energy is $U(x) = -k/x$. (a) Find its velocity as a function of position. (b) How long does it take the particle to reach the origin?

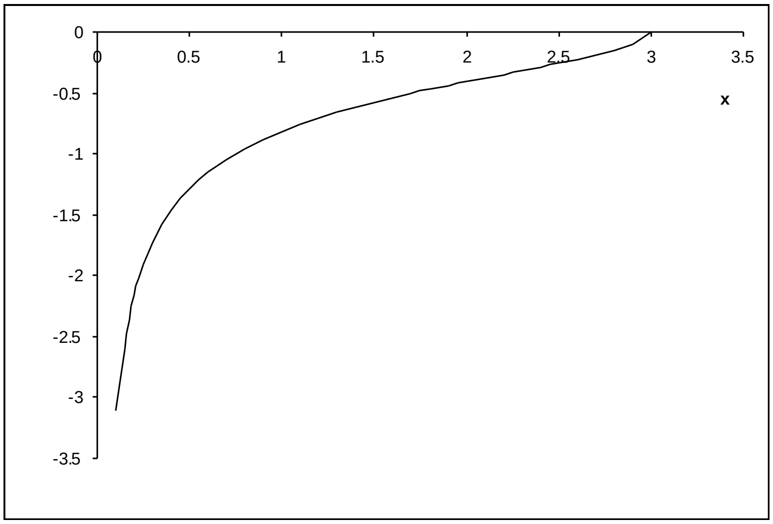
(a) At $x = b$, the kinetic energy is $T = 0$ so the total energy is $E = U(b) = -k/b$. Since energy is conserved:

$$E = -k/b = T + U(x) = \tfrac{1}{2} m\dot{x}^2(x) - k/x,$$

so taking the <u>negative</u> root because the potential attracts the particle toward the origin:

$$\dot{x}(x) = -\sqrt{\frac{2k}{m}\left(\frac{1}{x} - \frac{1}{b}\right)}.$$

An example of this is shown below.



(b) Since $\dot{x} = dx/dt$, the time required to move from b to 0 is:

$$\int_0^t dt = t = \int_b^0 \frac{dx}{\dot{x}(x)} = -\sqrt{\frac{m}{2k}} \int_b^0 \frac{dx}{\sqrt{1/x - 1/b}} = +\sqrt{\frac{mb}{2k}} \int_0^b \frac{\sqrt{x}\, dx}{\sqrt{b-x}}$$

Use the integral (from the front cover of the text):

$$\int \frac{\sqrt{y}\, dy}{\sqrt{1-y}} = \sin^{-1}\left(\sqrt{y}\right) - \sqrt{y(1-y)}$$

with the change of variables $x = by$ and $dx = b\, dy$. The integral for the time becomes:

$$t = \sqrt{\frac{mb}{2k}} \int_0^1 \frac{\sqrt{by}\, b\, dy}{\sqrt{b-by}} = \sqrt{\frac{mb^3}{2k}} \int_0^1 \frac{\sqrt{y}\, dy}{\sqrt{1-y}} = \sqrt{\frac{mb^3}{2k}} \left[\sin^{-1}\left(\sqrt{y}\right) - \sqrt{y(1-y)}\right]_0^1$$

$$t = \sqrt{\frac{mb^3}{2k}} \sin^{-1}(1) = \sqrt{\frac{mb^3}{2k}} \left(\frac{\pi}{2}\right)$$

$$t = \pi \sqrt{\frac{mb^3}{8k}}$$

Next two classes:

- Monday – Curvilinear 1-D Systems & Central Forces
- Wednesday – Multiparticle Systems