Phys 310 – Electronics
Lect 3a: AC Circuits (Phasors)

Fri. 1/29	**Ch 3, 4.5, 6.5:** AC Circuits	**Lab 2 Notebook**
Mon. 2/1 Wed. 2/3 Thurs. 2/4 Fri. 2/5	more of the same **Quiz** Ch 3**, Lab 3:** AC Circuits more of the same **Ch 4.2-.3, Ch 5:** Transformers, Diodes, & Power Supplies	**HW3**: Ch3 Pr 2,6,7*, 8*, 11,12,16 **Lab 3 Notebook**

Equipment
- O'Scope
- Fourier Transform box
- Speaker
- Necessary cables and connectors

Handout:
- Lab #3
- Practice with Phasors

Topics:
AC terminology for sinusoidal signals:
1. Sizes of voltages and currents: ANY of these can be used in Ohm's Law
   a. Peak value (amplitude)
   b. Peak-to-peak value (twice the amplitude)
   c. Root mean square (rms) – related to average power dissipated
2. Period, frequency, & angular frequency
3. Phase – leading and lagging

Fourier decomposition – only need to figure out response of circuits to sine waves

Reactance & Inductance – using Ohm's Law & Kirchoff's Rules for AC circuits

Phases of components - ELI the ICE man (emf is voltage)

Phasors – similar to vectors, represent size and phase of inductances

Calculations for various filters

Study List for Quiz #3:
1. Sine waves – amplitude, frequency, angular frequency, period, and phase.
2. Reactances and impedances of R, L, and C - complex number (j-operator) representation.
3. Using the phasor representation to calculate the magnitudes and phase angles of voltages and currents.

Equation List:

$|X_C| = 1/\omega C = 1/2\pi f C$ $\quad$ $Z_C = -j/\omega C = -j/2\pi f C \;\left[1/(\text{Hz}\cdot\text{F}) = 1\Omega\right]$ $\quad$ $j = \sqrt{-1}$

$|X_L| = \omega L = 2\pi f L$ $\quad$ $Z_L = j\omega L = j2\pi f L \;\left[1\,\text{Hz}\cdot\text{H} = 1\Omega\right]$ $\quad$ $\omega = 2\pi f$

$V = \sqrt{(V_{\text{real}})^2 + (V_{\text{imaginary}})^2}$ $\quad$ $\phi = \tan^{-1}(V_{\text{imaginary}}/V_{\text{real}})$ $\quad$ $\tilde{V} = V\angle\phi$

$\tilde{A}\times\tilde{B} = A\times B\angle(\phi_A + \phi_B)$ $\quad$ $\tilde{A}/\tilde{B} = A/B\angle(\phi_A - \phi_B)$ $\quad$ $\tilde{v} = \tilde{i}\tilde{Z}$

?Reactance, Impedance, and phasors.. what are the importance of these? They help us move into analyzing AC circuits. Then when we add in Fourier's theorem, we're really able to analyze circuits with any varying input signals. Think audio electronics – the input signals are definitely varying.?

Ch 3 AC Circuits I

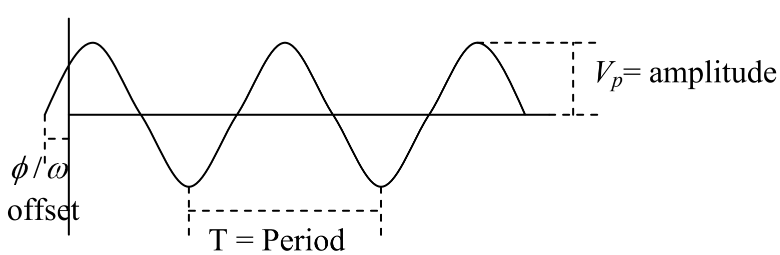
3.1 **Intro.** First we dealt with DC circuits, more specifically, circuits with constant voltages and currents. Next we considered transients – how circuits respond when the input voltage changes from one steady state to another. Now we're moving on to constantly oscillating input voltages; how do circuits respond to these?
Through this chapter we develop and work with tools for handling *sinusoidally* varying voltages. Now, that might seem of fairly limited use; however, the reading from chapter 4 points out that *any complicated signal* can be built out of sinusoidally varying ones of different amplitudes, frequencies, and phases. More about that later, but for now, if that's true, then the tools we'll develop here are actually of much broader utility for they can be applied to each sinusoidal 'component' of a more complicated signal.

3.2 **Sine.** You're no doubt quite familiar with this, but since we'll be dealing with it a good deal, it's worth a quick review. Say a voltage varies sinusoidally, i.e., as a function of time it looks like

$$v(t) = V_p \sin(\omega t + \phi)$$

where
V_p is the amplitude, a.k.a. peak value, of the voltage
ω *is* the angular frequency in radians / sec
it's related to the frequency and period via $\omega = 2\pi f = \frac{2\pi}{T}$
ϕ is the "phase" or 'offset' angle.

Here's how they all fit together on a plot



Could you go over the derivation for the root-mean-square current?

RMS

If you're probing a circuit with an oscilloscope, you can actually see the time varying signal, as we did in lab; however, if you're using a multimeter, you can't, or if you could, it would just be a bunch of flickering numbers and hard to make sense of. Quite often, it's handy to get *some* single, representative value from a multimeter (they sure are more compact and easier to carry around.) The obvious one is the amplitude. V_p; however, for more practical than theoretical reasons, multimeters don't give that. Instead, they give something that's known as the RMS, or Root Mean Square, value, where the Mean is taken over a period.

$$v_{rms} = \sqrt{\langle v(t) \rangle} = \sqrt{\frac{\int_o^T \left(V_p \sin(\omega t)\right)^2 dt}{T}} = \sqrt{\frac{V_p^2 \int_o^T \left(\sin(\omega t)\right)^2 dt}{T}} = \sqrt{\frac{V_p^2 \frac{1}{2} T}{T}} = \frac{V_p}{\sqrt{2}}$$

For seeing why the integral evaluates to this, recall that $\sin^2 + \cos^2 = 1$. Thus,

$$\int_o^T \left(\sin(\omega t)\right)^2 + \left(\cos(\omega t)\right)^2 dt = \int_o^T 1 dt = T$$

Draw pictures of $\sin^2$ and a picture of $\cos^2$ for a full period; clearly, they have the same area under their curves, so if their sum is 1T, then each one must individually contribute half of that, ½ T.

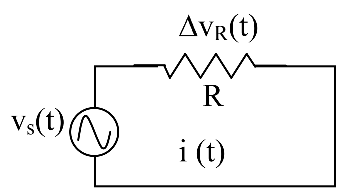
Similarly for a sinusoidally oscillating current.

$$i_{rms} = \frac{I_p}{\sqrt{2}}$$

Note: that ½ came because we were dealing with a very specific function. If the signal had a different functional behavior, it would have a different integral, and a different numeric factor would result (this will come up on a homework problem.)

AC with a Resistor

Let's treat a very simple circuit



If $i_s(t) = I_p \sin(\omega t)$, what's $\Delta v_R(t) = ?$

Ohm's Law: $\Delta v_R(t) = -i(t)R = -I_p \sin(\omega t) R = -\left(I_p R\right)\sin(\omega t)$

Clearly, the current and voltage *amplitudes* are related by:

$$\Delta V_{R.p} = I_p R \quad \text{or} \quad \frac{\Delta V_{R.p}}{I_p} = R$$

And we can write,

$$\Delta v_R(t) = -\Delta V_{R.p} \sin(\omega t)$$

For that matter, the root mean square values are related by

$$\Delta V_{R.rms} = I_{rms} R$$

Energy dissipation / Power.

An interesting side note is the energy dissipated by an AC circuit. As the current and voltage are constantly fluctuating, so must be the power, but we can at least speak of the *average* power, per period. Now energy's being dissipated by the resistor (when a lot of current passes through a resistor you can easily feel it warm up, so the energy is then dissipated just as from any other hot object.)

At any givne moment,

$$P(t) = i(t)\Delta v_R(t)$$

So averaged over one period,

$$\langle P\rangle = \frac{\int_0^T i(t)\Delta v_R(t)dt}{T} = \frac{\int_0^T I_p \sin(\omega t)(-\Delta V_R \sin(\omega t))dt}{T} = -I_p \Delta V_R \frac{\int_0^T \sin(\omega t)\sin(\omega t)dt}{T}$$

$$\langle P\rangle = -I_p \Delta V_R \frac{\int_0^T (\sin(\omega t))^2 dt}{T} = -I_p \Delta V_R \frac{T}{2T} = -\frac{I_p \Delta V_R}{2}$$

Conveniently, that's exactly what you get if you multiply the I_{rms} and ΔV_{rms}:

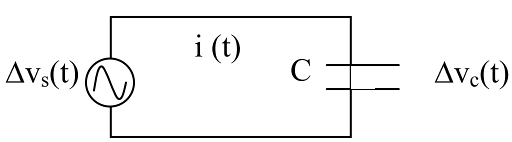
$$\langle P\rangle = -\frac{I_p \Delta V_R}{2} = -\frac{I_p}{\sqrt{2}}\frac{\Delta V_R}{\sqrt{2}} = -I_{rms}\Delta V_{rms}$$ (the sign just indicates that the energy is flowing *out*).

Calculating the impedance of components.
1) I dont understand impedance and it's equation that describes it. Z=R+jX what does each variable represent?

3.3 Reactance, Impedance, and Phasors

A single resistor circuit is easy, it get's a little more complicated if you've got a single capacitor or a single inductor, and much more complicated if you've got combinations. To handle these, we'll make some new definitions (Reactance and Impedance) and invoke some new mathematical tools (Phasors.)

Capacitive Reactance



Okay, again, say that $i(t) = I_p \sin(\omega t)$, what's $\Delta v_c(t) = ?$

$$\Delta v_C(t) = -\frac{q(t)}{C}$$

$$\Delta v_C(t) = -\frac{1}{C}\int i(t)dt$$

$$\Delta v_C(t) = -\frac{1}{C}\int I_p \sin(\omega t)dt$$

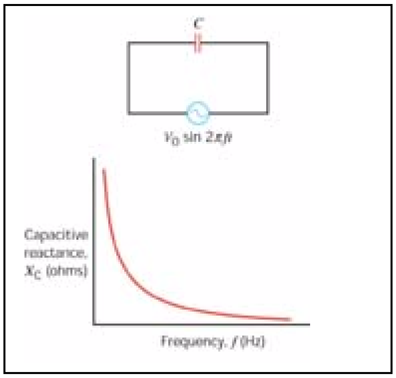
$$\Delta v_C(t) = \frac{1}{C\omega} I_p \cos(\omega t) = -\left(\frac{1}{C\omega} I_p\right)\sin(\omega t - \tfrac{\pi}{2})$$

$$\Delta v_C(t) = -\Delta V_{C.p} \sin(\omega t - \tfrac{\pi}{2})$$

(writing it this way will be useful later)

Now the amplitudes are related by $|X_c| \equiv \frac{1}{C\omega}$

$$\frac{\Delta V_{C.p}}{I_p} = \frac{1}{C\omega} \equiv |X_C|;$$ **Capacitive Reactance**



This factor then plays a similar role to that of resistance in determining how the circuit component reacts to having a current driven - what amplitude voltage is necessary to drive what strength current. This defines the "Capacitive Reactance." Depending on the frequency, the capacitor could require a very large or a very small voltage to drive the same current onto it. The higher the frequency, the less the capacitor impedes current flow (logically, it has less time to charge up, so it acquires less charge which then produce less field opposing current flow).

Energy Dissipation

Qualitative:

Think of the analogous system: you can put energy into a spring by compressing it, and you can recover that energy by letting the spring decompress; even over just the compression, no energy is *lost*, it's just converted from one form to another, and over one full cycle of compression and relaxation, you're back where you began – the exact same amount of energy back in the exact same form. The same is true for a capacitor – you trade the kinetic energy of moving charged particles for potential energy of stored charged particles / strong electric field, and then you set them moving again – no energy is lost by the capacitive action. (In point of fact, charges accelerate when you charge up or discharge a capacitor, thus *some* energy is lost to radiation).

Quantitative:

$$\langle P\rangle = \langle I(t)\Delta V_C(t)\rangle = \langle I_o \sin(\omega t)\cdot V_o \sin(\omega t - \tfrac{\pi}{2})\rangle = \langle I_o \sin(\omega t)\cdot V_o \cos(\omega t)\rangle$$

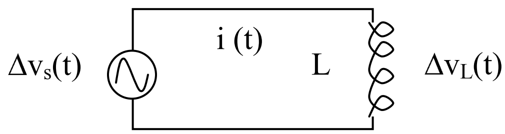
$$= I_o V_o \langle \cos(\omega t)\cdot \sin(\omega t)\rangle = 0$$

The 0 comes about because the "averaging" requires integrating cos*sin over one period and that comes to 0

$$\frac{\int_0^T \cos(\omega t)\cdot\sin(\omega t)dt}{T} = -\frac{1}{\omega}\frac{\int_{\cos(0)}^{\cos(\omega T)} \cos(\omega t)d\left(\cos(\omega t)\right)}{T} = -\frac{1}{\omega}\frac{\left(\cos(\omega T)-\cos(0)\right)}{T} = -\frac{1}{\omega}\frac{\left(1-1\right)}{T} = 0$$

Inductive Reactance

Similarly, if you have just an inductor, the circuit evaluates as follows



Ask them to do:

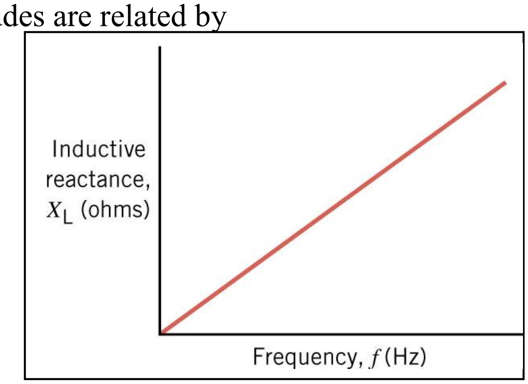
$$\Delta v_L(t) = -L\frac{di(t)}{dt} = -L\frac{d}{dt}I_p\sin(\omega t) = -L\omega I_p\cos(\omega t) = -(L\omega I_p)\sin(\omega t + \tfrac{\pi}{2})$$

$$\Delta v_L(t) = -\Delta V_{L.p}\sin(\omega t + \tfrac{\pi}{2})$$

(writing it this way will be useful later)

So the voltage and current amplitudes are related by

$$\frac{\Delta V_{L.p}}{I_p} = L\omega \equiv |X_L|$$



Now we have the inductive reactance $|X_L| = L\omega$ determining how much voltage is required for a given current.

Energy Dissipation

The math is essentially the same as for the capacitor – no energy is lost due to the inductive effect. (Again, radiation does occur when currents speed up or slow down, so some energy is radiated away).

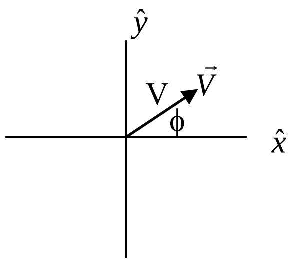
Complex circuits (analyzed in the complex plane)

This is all good and well, and rather manageable for single-component circuits, but what if they're all mixed up? What if you have a circuit with a capacitor and an inductor, or an inductor and a resistor? Or all three? One *can* actually slog through the math of it using the tools we already have, but there's a slicker way. Mind you, it's a little more mathematically abstract, but when it comes right down to it, it's easier to do.

Magnitude & Phase affects – calls for 2-D vectors. Each component affects the voltage in two ways: it affects the *amplitude* and it effects the *phase angle* (notice that the voltage across a capacitor or an inductor is out of phase with the current.) That sounds like something that can be represented by a 2-D vector in some abstract mathematical space. Of course, vectors are cumbersome – you've got two components to deal with, you've got to be careful how you multiply them and add them…Now, a really concise way of dealing with 2-D vectors is going to the Complex plane. The plane may be complex and novel, but the math is mostly simple and familiar.

You're familiar with working in the regular 2-D plane. Say (where y and x are just mathematical coordinates, not real spatial coordinates)

Off to Mathland: introducing Phasors and the Complex plane



In terms of these coordinates, we might write

$$\vec{V} = V_x\hat{x} + V_y\hat{y} = V\cos\phi\hat{x} + V\sin\phi\hat{y}.$$

For that matter, now's a good time to recall two useful relations:

$$V = \sqrt{V_x^2 + V_y^2}$$

$$\phi = \tan^{-1}\left(\frac{V_y}{V_x}\right)$$

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Q:  2)I'm not familiar with their polar coordinate notation.  p. 51 has a bunch
of examples.  What does the backslash followed by underlined angles mean?
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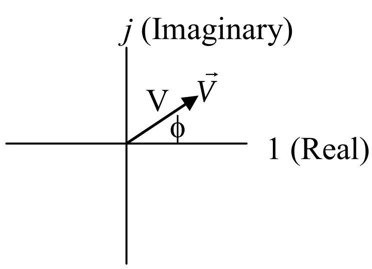
> You may be familiar with the notation $\vec{V} = (V, \phi)$ for a 2-D vector. The book uses the notation $\vec{V} = V\angle\phi$ to mean the same thing. Since most of us aren't familiar with that, I won't personally be using it, but you should feel free to.

Even handier than the real x-y plane is the Complex real-imaginary plane.



In place of $\hat{x}$ we just have 1, and in place of $\hat{y}$ we have *j* where $j = \sqrt{-1}$ (the other standard symbol, *i,* would too easily get confused with current in this context.)
In terms of *these* coordinates, we might write

$$\vec{V} = V_R + V_I j = V\cos\phi + Vj\sin\phi = V(\cos\phi + j\sin\phi).$$

We still have the magnitude and phase relations,

$$V = \sqrt{V_R^2 + V_I^2}$$

$$\phi = \tan^{-1}\left(\frac{V_I}{V_R}\right)$$

But we also have

$\cos\phi + j\sin\phi = e^{j\phi}$ (write out their Taylor Series' and you'll see it's so).

So,

$$\vec{V} = V(\cos\phi + j\sin\phi) = Ve^{j\phi}.$$

With this definition of a phasor, the algebraic rules easily relate to rules you're familiar with. Say,

$$\vec{A} = A_R + A_I j,\ \vec{B} = B_R + B_I j$$

Then,

$$\vec{A} + \vec{B} = (A_R + A_I j) + (B_R + B_I j) = (A_R + B_R) + (A_I + B_I) j$$

Alternatively, for multiplication and division, it's easiest to use the other representation:

$$\vec{A}\cdot\vec{B} = |A|e^{j\phi_A}|B|e^{j\phi_B} = |A||B|e^{j\phi_A}e^{j\phi_B} = |A||B|e^{j(\phi_A+\phi_B)}$$

Note: you can also multiply this way

$$\vec{A}\cdot\vec{B} = (A_R + A_I j)\cdot(B_R + B_I j) = (A_R B_R + jjA_I B_I + jA_R B_I + jA_I B_R)$$
$$= (A_R B_R - A_I B_I) + j(A_R B_I + A_I B_R)$$

The negative sign comes because $jj = \sqrt{-1}\sqrt{-1} = -1$. If you find the magnitude and angle of this, it's the same as you get the other way.

For next time: work on the Practice with Phasors handout. Hand out.

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pg 52 Low/high pass filter derivation is a little tricky to follow

For RLC resonant circuts, what is the Z vector?
3)What is the quality factor, Q? how does it relate to a circuit? – we'll get to
next time – it quantifies the sharpness of a peak / trough (?)
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Back from Mathland: using Phasors and the Complex plane for AC circuits

Let's see how we can represent the affects of Resistors, Inductors, and Capacitors on oscillating currents and voltages using this mathematical scheme. Unfortunately, this brings in one more bit of vocabulary. The complex-plane vectors that represent the effects of resistors, capacitors, and inductors on AC currents are known as their ***impedances,*** and I'll denote them $\vec{Z}_R$, $\vec{Z}_C$, and $\vec{Z}_L$ such that, the *magnitude* of Z is the ratio

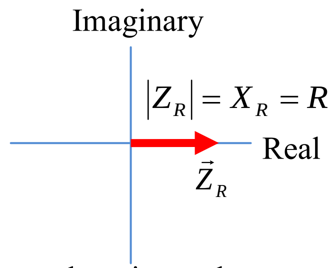
$$|Z| = \frac{\Delta V_p}{I_p} = X,$$

i.e., the amplitude of this abstract quantity is the very real *reactance*.

And the *angle* of Z is the voltage's phase relative to what it would be if we had a simple resistor.

Resistor

Say we're using a current of $i(t) = I_p \sin(\omega t)$, then the voltage across a resistor is simply related to the current through it by



$$\Delta v_R(t) = -RI_p \sin(\omega t)$$

So,

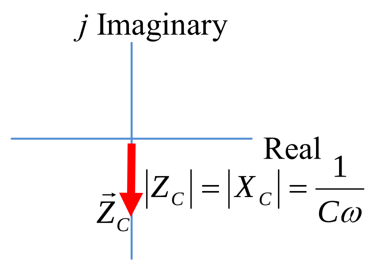
$$\frac{\Delta V_{R.p}}{I_p} = R\text{, and,}$$

since the resistor defines our reference, there is no phase.

$\vec{Z}_R = R$ (implicit direction is along the Real axis)

Capacitors

Again, using a current of $i(t) = I_p \sin(\omega t)$, the voltage across a capacitor is related to the current flowing to it by



$$\Delta v_C(t) = -\frac{1}{C\omega} I_p \sin(\omega t - \tfrac{\pi}{2})$$

Amplitude: $\frac{\Delta V_{C.p}}{I_p} = |X_C| = \frac{1}{C\omega}$

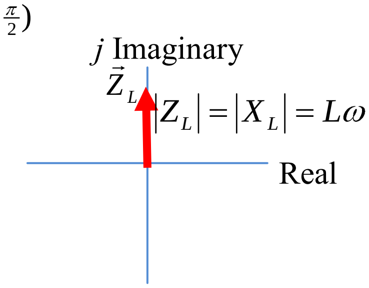
Phase: $\phi = -\frac{\pi}{2}$

$$\vec{Z}_C = j\frac{-1}{C\omega} = \frac{1}{C\omega}e^{-j\frac{\pi}{2}} = \frac{1}{C\omega}\angle -\tfrac{\pi}{2}$$

(the factor of j plays the role of $\hat{y}$ to indicate the direction is along the Imaginary axis)

Inductors

Again, using a current of $i(t) = I_p \sin(\omega t)$, the voltage across an inductor is related to the current flowing through it by $\Delta v_L(t) = -L\omega I_p \sin(\omega t + \frac{\pi}{2})$



Amplitude: $\frac{\Delta V_L}{I_p} = L\omega = |X_L|$

Phase effect: $\phi = \frac{\pi}{2}$

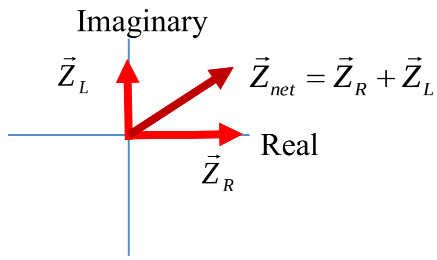
$$\vec{Z}_L = jL\omega = L\omega e^{j\frac{\pi}{2}} = L\omega\angle\tfrac{\pi}{2}$$

(the factor of j plays the roll of $\hat{y}$ to indicate the direction is along the Imaginary axis)

3.4 AC Analysis of RC Circuit

Combinations

Here's where this kind of notation and math becomes *useful.* If you have a circuit with, say an inductor *and* a resistor in series, then the net impedance would simply be the vector sum of the two:



Next up: Putting it all to use.

I'll do two examples. The first one I'll fully narrate, so you can follow what's going on, but it'll look much longer than it needs to. The second one, I'll just crunch through, so you'll get a better sense of how much work is involved.