The Planar Area Location/Layout Problem*

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Abstract

This paper considers the problem of placing a single rectangular *generalized congested region* (GCR) (closed and bounded region in $\Re^2$ which prohibits facility location but allows travel through at a penalty) of given area but unknown dimensions in the presence of other rectangular GCRs. Due to the variable dimensions of the new GCR, the unique classification of user-user and user-server flows through gridlines becomes challenging. We overcome this difficulty and provide solution methodologies for two variants of the problem: (i) new GCR's server located on its boundary, or (ii) at its centroid. These are shown to be polynomially bounded in the number of GCRs.

Keywords: Single Facility Location/Layout, Generalized Congested Regions, Rectilinear Distance Metric.

1 Introduction

Facility layout and facility location are critical components in the overall problem of facilities design. However, the facility layout problem and the facility location problem have grown into different areas of interest even though they are closely related. Facility layout and facility location both deal with the location of new facilities in a region where there are existing facilities. In the facility location problem, facilities are typically infinitesimal. However, in facility layout, new constraints determining the area requirements and locational restrictions have to be considered because the facilities have finite sizes. In essence, the facility layout problem can be viewed as an *area location/layout/placement problem*.

Restricted location problems are a class of location problems in which the minimum travel distance between two points in $\Re^2$ is increased by certain predescribed areas which obstruct travel. **Barriers**, **forbidden regions** and **generalized congested regions (GCR)** (prohibit facility location but allow travel through at a penalty: for example, assembly areas, finishing areas on the shop floor, through which travel is permitted but is slowed down due to congestion) are examples of restrictions. Barriers (prohibit travel through) and forbidden regions (do not penalize travel through) are special cases of GCRs. Restricted location problems have been studied by [4], [1], [7], [3], [2] and others. Most of these works determine the optimal location of an infinitesimal new facility such that the (i) total weighted travel distance from the new facility to the existing facilities (minisum or median objective), or (ii) maximum weighted travel distance from the new facility to the existing facilities (minimax or center objective) in the presence of restricted regions is minimized. In contrast, the objective of the facility layout problem is to minimize the total weighted distance between entities, in which the weights indicate material flow volumes or adjacency priorities. The facility layout problem is well studied in literature and practice ([8], [5], [6]).

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In this paper, we develop the preliminary results for a new approach to the facility layout problem. We cast the facility layout problem as a facility location problem, in which GCRs pose restrictions to travel. More specifically, we consider the problem of placing a new facility, which itself is a GCR, in the presence of existing facilities that are treated as GCRs to travel. The area of the new GCR to be placed is known but its exact dimensions are unknown. Assembly areas and finishing areas in a manufacturing facility could be considered to be GCRs through which travel is permitted but is penalized due to congestion slowdown. Also the exact dimensions of such areas may vary but their shapes can be approximated by rectangles. In a way, this paper may be viewed as an extension of the work by Savas *et al.* [10] in which the new facility is arbitrarily shaped but has a fixed contour. Also the restriction in [10] comes in the form of barriers, as compared to GCRs in our work.

The remainder of this paper is organized as follows. In §2, we formally define the problem, introduce some notation and provide preliminaries. Our solution methodology described in §3, is split into two sub-sections: when the new GCR placement does not intersect any gridline in §3.2 and when the new GCR intersects at least one gridline in §3.3. The complexity of the solution methodologies is analyzed in §4 and conclusions and directions for future research are outlined in §5.

2 Problem Description

2.1 Problem Statement

There exists a finite number of rectangular ***generalized congested regions (GCRs)*** with edges parallel to the travel axes. The additional cost per unit distance is called the congestion factor of the GCR and is denoted by α, $0 \leq \alpha < \infty$. Each GCR has one or multiple **users** (input/output (I/O) points) through which it communicates with other GCRs. The users are located inside the GCRs or on their boundaries.

A new rectangular GCR with its edges parallel to the travel axes is to be placed in the presence of existing GCRs. The congestion factor and area of the new GCR are known, but not its exact dimensions. The new GCR communicates with the users of the existing GCRs through a single **server**. The server acts like an I/O point like other users. We consider the following two versions of the problem:

- when the server is located on the boundary of the new GCR but its exact location is unknown, referred to as the "unknown boundary server problem", and
- when the server location is known *a priori* and is assumed to be at the centroid of the new GCR, referred to as the "centroid server problem".

In both versions of the problem, there exist flows between:

- pairs of users of existing GCRs, termed as the *user-user interaction*. This interaction takes place along least cost paths [2] between user pairs.
- an existing user and the server of the new GCR, termed as the *user-server interaction*. This interaction also takes place along least cost paths between users and the server.

The *planar area location/layout problem* is to determine the exact dimensions (specified by the length) of a new rectangular GCR and its optimal location (specified by the location of its server and the location of its top left corner) such that the new GCR does not overlap with existing GCRs and the sum of the user-server and user-user interaction is minimized.

2.2 Definitions and Notations

Since the new GCR is a finite-sized entity, the coordinates of its server alone cannot convey full information on its placement in $\Re^2$. Hence we define $p = [X, E_4(\overline{B}), l]$ to be the *location-dimension vector* of the new GCR. Here $X = (x, y)$ and $E_4(\overline{B})$ represent the location of the server of the new GCR and its top left corner respectively. l represents the length of the new GCR and is measured along the x axis. Note that l x $b = A$, where A denotes the area of the new GCR and is a known parameter and b is its width.

Let D denote the set of all users. For a given facility placement p, the total weighted travel distance between users and the facility (user-server interaction) is $J(p)$ and correspondingly between all users (user-user interaction) is $K(p)$.

$$J(p) + K(p) = \sum_{j \in D} u_j \; d_p(j, X) + \sum_{i \in D} \sum_{j \in D} w_{ij} \; d_p(i, j).$$

Here u_j and w_{ij} represent the amount of user-server interaction (between user j and server X) and user-user interaction (between users i and j) respectively. $d_p(A, B)$ represents the length of the least cost path between points A and B for placement p of the new GCR. The planar area location/layout problem is to determine the exact dimensions of the new rectangular GCR and its optimal placement p such that $J(p^*) + K(p^*) \leq J(p) + K(p), \; \forall p \in F$.

For the **unknown boundary server problem**, there are five continuous variables, the coordinates of $E_4(\overline{B})$ and X, and l. For the **centroid server problem**, we have three continuous variables since the coordinates of X are known once $E_4(\overline{B})$ and l are specified. For either situation, it turns out that the objective function is non-convex and non-concave and hence the use of generalized gradient methods could lead to a local optimal solution. We are able, however, to obtain a global optimal solution by the following line of attack. We divide the feasible region associated with $E_4(\overline{B})$ into subregions where the objective function is concave and thus a finite set of candidate values for $E_4(\overline{B})$ and X are obtained. For each such candidate set of values of $E_4(\overline{B})$ and X, we then optimize l. This allows us to conclude that the solution methodology is polynomial in the number of existing GCRs.

2.3 Preliminaries: Grid Construction and Cell Formation

In order to develop our analysis, we first describe the grid construction procedure that helps to identify the least cost path between two points in the presence of GCRs. A grid is constructed by passing horizontal and vertical lines through the vertices of each GCR and its users. The lines intersect the other GCRs and pass through until they terminate at the smallest bounding rectangle (smallest rectangle that encloses all existing GCRs with its edges parallel to the travel axes). For simplicity in presentation, we shall refer to these lines as "gridlines" henceforth. The GCRs along with the gridlines divide the feasible region F into a number of regions, called ***cells***. Since the GCRs are rectangular and due to the way the grid is constructed, all cells generated are also rectangular with their edges parallel to the travel axes. We denote the corners of a cell C as $E_k(C)$, $k = 1, 2, 3, 4$ starting from the bottom left corner and labeling them in the counter clockwise direction. Theorem 3.1 of Sarkar *et al.* [9] proves that at least one least cost path between two points in the presence of rectangular GCRs will coincide with segments of the grid obtained by following the procedure described earlier. We conclude this section by stating the following lemma without proof. The lemma is central to our solution methodology.

Lemma 2.3.1. *A feasible rectilinear least cost path from a user outside a cell C to an infinitesimal point located inside C passes through a cell corner $E_k(C), k = 1, 2, 3, 4$.*

Due to Lemma 2.3.1, the least cost path between a user and a point inside a cell can be split into two parts: from the user to a cell corner and then from the cell corner to the point.

3 Solution Methodology

3.1 Problem Classification

In our problem, the area of the new GCR is a known parameter. If the area of the new GCR exceeds the area of a cell C, then the new GCR cannot be fully contained in the cell. Hence it will intersect gridlines thereby interrupting the flows between user pairs. However if the area of the new GCR is less than the area of a cell, then it can be fully contained in the cell. In this scenario, the user-user interaction will remain unaffected. This observation motivates us to study the problem for two cases, when:

1. the new GCR does not intersect any gridline (in §3.2), and

2. the new GCR intersects at least one gridline (in §3.3).

3.2 The new GCR placement does not intersect any gridline

First we analyze the **unknown boundary server problem** with the aid of Lemma 3.2.1.

Lemma 3.2.1. *When the new GCR is fully contained in a cell C, its server X coincides with a cell corner $E_k(C)$, $k = 1, 2, 3, 4$.*

The proof follows from the fact that the user-user interaction $K(p)$ remains unchanged in this case (since no gridlines are intersected) whereas the user-server interaction $J(p)$ is concave in cell C. The lemma follows because $J(p)+K(p)$ is hence concave in cell C. The new GCR can have any dimensions l and b that satisfy its area requirement.

For the **centroid server problem**, candidate optimal locations of the new GCR's corners are identified by Lemma 3.2.2 which is stated without proof as follows.

Lemma 3.2.2. *When the new GCR is fully contained in a cell C, the optimal placement of the new GCR is such that one of its corners $E_k(\overline{B})$ coincides with a corner $E_k(C)$, $k = 1, 2, 3, 4$, of cell C.*

The outline of the proof is similar to that of Lemma 3.2.1. The three-variable problem thus reduces to a one-variable problem in l. The objective function $J(p) + K(p)$ can be expressed in terms of l, of the form $\psi_1 l + \frac{\psi_2}{l} + \psi_3$, where ψ_1, ψ_2 and ψ_3 are constants in terms of A, α, u_{ik}'s which are all known parameters. Depending upon the values of ψ_1 and ψ_2, $J(p) + K(p)$ is either (i) convex in l, or (ii) concave in l, or (iii) strictly increasing in l, or (iv) strictly decreasing in l. For each of the cases, a unique minima for l can be obtained.

3.3 The new GCR placement intersects at least one gridline

When the area of the new GCR exceeds the area of a cell, the new GCR cannot be contained fully in the cell. Hence it will intersect gridlines possibly disrupting the flows between users as the new GCR may interfere with the least cost path between existing users. Hence the least cost path between two users may have to travel through the new GCR or bypass it, increasing the cost of the path in either case. Hence it is critical to specify/identify the gridlines which the new GCR intersects for a particular placement p.

To this end, let $\mathcal{Q}(l)$ denote the set of placements of the new GCR such that when $E_4(\overline{B}) \in \mathcal{Q}(l)$, the new GCR will always intersect the same gridlines for a particular length l of the new GCR. The idea is illustrated in Figure 1 in which the new GCR having length l intersects vertical gridlines v_1, v_2, v_3, v_4 and horizontal gridlines h_1, h_2, h_3. The set $\mathcal{Q}(l)$ obtained for this length l of the new GCR is illustrated by the dotted rectangle.

We establish that $\mathcal{Q}(l)$s are rectangular with their edges parallel to the travel axes. Note that the corners of $\mathcal{Q}(l)$ can be expressed as functions of l. However it does not suffice to define $\mathcal{Q}(l)$s. Unlike barriers, one may wish to pass through or bypass GCRs depending on (i) the congestion factor of the GCR, (ii) the dimension of the GCR, i.e., the

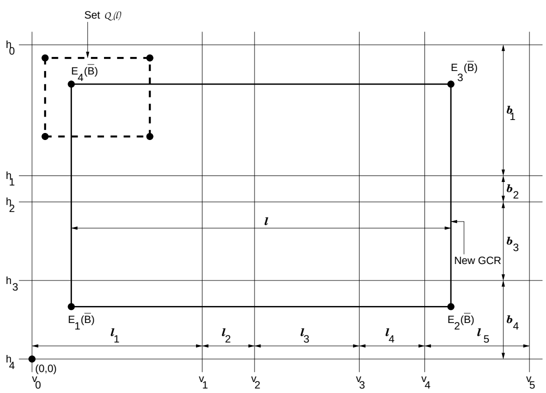

Figure 1: Construction of set $\mathcal{Q}(l)$

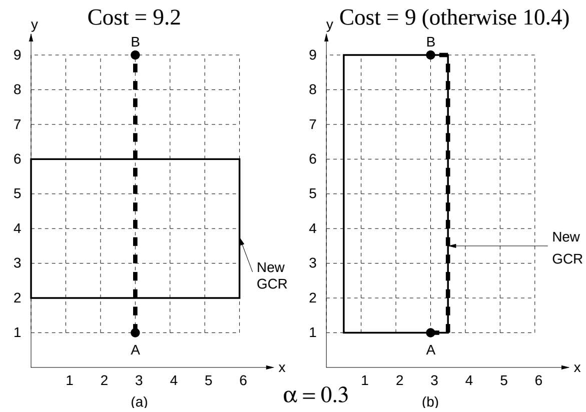

Figure 2: Effect of dimension change

distance traveled inside the GCR, and (iii) the distance traveled to bypass the GCR. In our problem, when the new GCR intersects gridlines, the flows may choose to (i) pass through, or (ii) left bypass, or (iii) right bypass the new GCR. Here "left bypass" and "right bypass" signify flows that bypass a GCR along its left or right edge respectively. Due to the variable dimensions (i.e., varying l) of the new GCR, unique classification of the flows as above becomes challenging. The difficulty is illustrated in Figure 2 in which a new GCR of area 24 units is to be placed. When the new GCR is a 6x4 rectangle, the least cost path between A and B (dotted line in Figure 2a) passes through the GCR at a cost of 9.2 units. However when the new GCR is a 8x3 rectangle, as illustrated in Figure 2b, the least cost path between A and B bypasses the GCR along its right edge at a cost of 9 units. Had it passed through the new GCR, the cost would be 10.4 units.

Hence we partition set $\mathcal{Q}(l)$s by Equal Travel Time Partitions $(ETTP)$. The significance of an $ETTP$ is as follows: for a feasible placement $p \in \mathcal{Q}(l)$, if $E_4(\overline{B})$ lies on the $ETTP$, the least cost path between a pair of users may pass through or left/right bypass the new GCR. $ETTP$s for vertical (horizontal) gridlines partition a $\mathcal{Q}(l)$ vertically (horizontally) into smaller rectangles $\mathcal{R}^{\mathcal{Q}(l)}$. In each such $\mathcal{R}^{\mathcal{Q}(l)}$, the flow classifications are unique. Hence distances (between user pairs or between a user and the server) can be accurately measured. Also each gridline can possibly generate two $ETTP$s, one each for left and right bypass. Note that the corners of $\mathcal{R}^{\mathcal{Q}(l)}$ are also functions of l. We now state the following result without proof.

Lemma 3.3.1. *For a given l, candidate optimal locations of $E_4(\overline{B})$ are the corners of the sets $\mathcal{R}^{\mathcal{Q}(l)}$.*

The previous discussion applies to both the unknown boundary server problem and centroid server problem. Potential candidate server locations in case of the unknown boundary server problem are now identified by Lemma 3.3.2.

Lemma 3.3.2. *The candidate points for optimal server location are (i) corners of the new GCR, and (ii) points of intersection of gridlines with edges of the new GCR.*

Again, all candidate server locations are functions of l. For each candidate location of $E_4(\overline{B})$ and X, $J(p)+K(p)$ can be expressed as a function of l (the five variable problem reduces to a one variable problem) of the form $\lambda_1 l + \frac{\lambda_2}{l} + \lambda_3$, where λ_1, λ_2 and λ_3 are constants in terms of A, α, u_i, w_{ij} and the coordinates of the existing users. Alike §3.2, properties of the objective function can be established depending on the values of λ_1 and λ_2 and a unique minima for l can be obtained. The only difference in the centroid server problem is the centroid location of the server, whose coordinates can again be expressed in terms of l.

4 Solution Complexity

Our solution methodology to the area location/layout problem is based on evaluating the user-server interaction $J(p)$ at the cell corners for both versions of our problem, when the new GCR does not intersect gridlines. When the new GCR intersects gridlines, our analysis is based on the construction of $ETTP$s for set $\mathcal{Q}(l)$s. The number of cell corners, set $\mathcal{Q}(l)$s, rectangles $\mathcal{R}^{\mathcal{Q}(l)}$s and candidate locations of $E_4(\overline{B})$ and X are $O(N^2)$, $O(N^4)$, $O(N^6)$, $O(N^6)$ and $O(N^7)$ respectively, where N is the number of existing GCRs. Hence the overall solution procedure is polynomial in the number of existing GCRs. It is pertinent to note here that the construction of $ETTP$s for each intercepted gridline can also be performed in polynomial time.

5 Conclusions and Future Research

The novelty of this work is placing an area of unknown dimensions rather than a fixed area shape, in the presence of restrictions. A non-traditional approach is proposed in which the facility layout problem is cast as a facility location problem. Hence the concepts and results developed apply to restricted facility layout and location problems alike and may be extended to study more general location/layout problems. For example, the multiple server case for a single GCR is an interesting open problem. Firstly, we may study the case in which the number of servers is known *a priori*. Secondly, with fixed costs for opening servers and variable costs for using them, the problem to determine the optimal number of servers to be opened can be studied.

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References

[1] R. Batta, A. Ghose, and U. Palekar. Locating facilities on the manhattan metric with arbitrarily shaped barriers and convex forbidden regions. *Transportation Science*, 23(1):26–36, 1989.

[2] S.E. Butt and T.M. Cavalier. Facility location in the presence of congested regions with the rectilinear distance metric. *Socio-Economic Planning Sciences*, 31(2):103–113, 1997.

[3] P. M. Dearing, H. W. Hamacher, and K. Klamroth. Dominating sets for rectilinear center location problems with polyhedral barriers. *Naval Research Logistics*, 49(7):647–665, October 2002.

[4] R.C. Larson and G. Sadiq. Facility locations with the manhattan metric in the presence of barriers to travel. *Operations Research*, 31(4):652–669, January 1983.

[5] R.M. Meller and K.-Y. Gau. The facility layout problem: Recent trends and emerging perspectives. *Journal of Manufacturing Systems*, 15(5):351–366, 1996.

[6] B. Montreuil. A modeling framework for integrating layout design and flow network design. In *Proceedings of the Material Handling Research Colloquium, Hebron, KY, USA*, pages 43–58, 1990.

[7] P. Nandikonda, R. Batta, and R. Nagi. The weighted 1-center problem with arbitrary shaped barriers. *Accepted to Annals of Operations Research*, 2002.

[8] L. F. McGinnis R. L. Francis and J. A. White. *Facility Layout and Location: An Analytical Approach*. Prentice Hall, Englewood Cliffs, NJ, 1992.

[9] A. Sarkar, R. Batta, and R. Nagi. Commentary on ‘facility location in the presence of congested regions with the rectilinear distance metric’. *accepted to Socio-Economic Planning Sciences*, (corrected proof available online at http://authors.elsevier.com/trackpaper.html or http://www.eng.buffalo.edu/∼nagi/papers/crpaper.pdf), 2003.

[10] S. Savas, R. Batta, and R. Nagi. Finite-size facility placement in the presence of barriers to rectilinear travel. *Operations Research*, 50(6):1018–1031, 2002.